

Non-metric Quintessence



Master of Science thesis by
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Abstract

The existence of dark energy in the universe is hypothesized to explain the accelerated expansion of today's universe and the inflationary expansion of very early universe; accelerated expansion of today's universe is proved by observational data gathered from ground-based and space telescopes, from type Ia supernovas; and inflation of the early universe is assumed to solve issues of the Big Bang theory, for instance, flatness problem, and isotropy and homogeneity of the universe in large scale.

There are different models of dark energy; the first and simplest one, is cosmological constant (or vacuum energy), that is constant all the time and space; the others are quintessential models which are dynamics, evolving and they can be inhomogeneous. In this thesis, two different quintessential models of dark energy, scalar field and non-metric quintessence, have been studied.

Chapter one of this thesis is a brief review of general relativity and cosmology; in this chapter the need for a field which provides negative pressure and accelerates the expansion of our universe is discussed.

Chapter two, is the study of the equations of motion of scalar field with exponential potential. In this chapter I've studied the equilibrium points of the phase plots found from the dynamics of dimensionless variables of the scalar field. Then I've solved the equations of the evolution of a universe, filled with scalar field and background fluids, numerically. Plots of the evolution of density parameters found from this model show that this model with exponential potential, is not consistent with our universe.

In chapter three, equations of motion of a two-field system (scalar matter field with a power-law potential and graviscalar field), are derived from the lagrangian defining non-metric quintessence. I've found the phase plots of the system, and solved the equations of the evolution of a universe, filled with this two-field and background fluids (dark matter, baryonic matter and radiation), numerically. plots of the evolution of density parameters, are very compatible with the real picture of the history of the universe; and today's values of density parameters and Hubble parameter, age of the universe and the size of particle horizon, found from this model, are very close to their measured values (and sometimes the same as measured values). Although this model is very compatible with our universe, it doesn't explain inflation; and two-field system is in fact one scalar matter field.

In order to explain the inflation of the early universe, I decided to define a hybrid potential of scalar matter field, that is discussed very briefly in chapter four. Hybrid potential is a combination of exponential (gaussian) and power-law potential; its exponential part dominates early universe and it can explain inflation, and power-law part of the hybrid potential is a good

alternative for dark energy. The plots found from the numerical solution of the system of equations of dimensionless variables, are very compatible with the real history of our universe; and today's values of density parameters of dark energy, dark matter, baryonic matter, and Hubble parameter, and the age and size of our universe, found from this model, are very close to (and some times the same as) measured values. I found more compatible answers assuming that radiation dominates our universe at a very early stage and later on, exponential potential dominates and causes rapid expansion of the early universe.

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Chapter 1

Introduction

1.1 Reviewing of General Relativity

1.1.1 Metric Tensor

In Riemannian spaces, R_N , different points can communicate to each other and it is possible to write an equation which describes the relation between two given points' properties; this equation is called metric:

$$dS^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \quad (1.1)$$

where dS^2 or line element is the square of space-time interval between two neighbour points; dS^2 is a scalar and invariant under coordinate transformation. Defined in a coordinate system, dx^μ are the components of a contravariant vector that connects these two points. In addition to the differences between components, any displacement between two points is dependent on the positions of them in the coordinate system, through the function $g_{\mu\nu}(x)$, which is a covariant tensor of rank-2, called the metric tensor. In a Cartesian (rectangular) coordinate system because of the homogeneity, displacement between two points is independent of their components; and for a rectangular coordinate system build in Minkowski space-time, metric is defined as :

$$dS^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu = c^2 dt^2 - \sum_i (d\xi^i)^2 \quad (1.2)$$

where ξ^μ are the space-time rectangular coordinate components and ξ^i are only spatial parts of them; c is the speed of light; $\eta_{\mu\nu}$ is the metric tensor for this case and its matrix representation is:

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (1.3)$$

The components of $g_{\mu\nu}$ in a curved coordinate system can be found through coordinate transformation from rectangular coordinate system, $\eta_{\mu\nu}$, to the curved one; but rectangular coordinate system exists only in flat space-time (Minkowski space-time). It is impossible to construct rectangular coordinate system in a curved space-time; in the other words, curvature is obstruction to move from a curved coordinate to a rectangular one; but it is possible to do that locally if space-time is locally flat. Metric transformation from a rectangular coordinate system, ξ , defined locally in a point of a curved space-time, to a curved coordinate system, X , can be written as:

$$dS^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial X^\mu} \frac{\partial \xi^\beta}{\partial X^\nu} dX^\mu dX^\nu \implies$$

$$g_{\mu\nu}(X) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial X^\mu} \frac{\partial \xi^\beta}{\partial X^\nu} \quad (1.4)$$

So we can find the local values of metric tensor; but it is impossible to find its global value, except when the local curvature is the same at all points of the space-time.

Three important properties of metric tensor:

- $g_{\mu\nu}$ is symmetric.
- $\det(g_{\mu\nu}) \neq 0 \implies$ inverse matrix $g^{\mu\nu}$ exists, that is the contravariant form of metric tensor; so we have:

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \quad (1.5)$$

- metric tensors can be used to lowering or raising indices.

1.1.2 Equations of Motion

In a comoving rectangular coordinate system attached to a particle, the velocity of the particle is:

$$\hat{u}^\mu = \frac{d\xi^\mu}{d\tau} = \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (1.6)$$

Where τ is the proper time, which is a parameter along space-time curve of particle; so \hat{u}^μ can be transformed into a curved coordinates through expansion on curved coordinates components:

$$\hat{u}^\mu = \frac{d\xi^\mu}{d\tau} = \frac{\partial \xi^\mu}{\partial X^\nu} \frac{dX^\nu}{d\tau} \implies$$

$$\hat{u}^\mu = \frac{\partial \xi^\mu}{\partial X^\nu} u^\nu \quad (1.7)$$

where $u^\nu = \frac{dX^\nu}{d\tau}$ is the four-velocity of particle in curved coordinate system(X).

\hat{u}^μ is constant, so $\frac{d\hat{u}^\mu}{d\tau} = 0$, and we have [1]:

$$\begin{aligned} \frac{d}{d\tau} \left(\frac{\partial \xi^\mu}{\partial X^\nu} \frac{dX^\nu}{d\tau} \right) &= 0 \implies \\ \frac{d^2 X^\alpha}{d\tau^2} + \Gamma_{\mu\nu}^\alpha \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} &= 0 \end{aligned} \quad (1.8)$$

where,

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial X^\nu} + \frac{\partial g_{\beta\nu}}{\partial X^\mu} - \frac{\partial g_{\mu\nu}}{\partial X^\beta} \right) \quad (1.9)$$

are the Christoffel symbols; they describe the changes of the metric tensor under motion between different points of a coordinate system; they vanish in rectangular coordinate system. Christoffel symbols are in fact objects that give us covariant derivatives of tensors; the term covariant here refers to covariant under coordinate transformation; so covariant derivative of a tensor is a tensor. Ordinary derivatives of tensors do not transform tensorially, so they are not tensors. By writing coordinate transformation of vector A^μ , it is easy to show that its covariant derivative is:

$$A_{;\lambda}^\mu = A_{,\lambda}^\mu + \Gamma_{\nu\lambda}^\mu A^\nu$$

where $A_{,\lambda}^\mu = \frac{\partial A^\mu}{\partial X^\lambda}$ is the ordinary derivative of A^μ in the X coordinate system.

Equations(1.8) are equations of motion and they describe geodesics (extremal curves) in Riemannian space-time, where there is at least one geodesic in order to move from a point to the another one; in fact a geodesic is the trajectory of a freely moving particle; that is straight a line in Minkowski space-time. We can show that the equations (1.8) represent covariant derivative of vector $\frac{dX^\alpha}{d\tau}$ by substituting $\frac{d}{d\tau} = \frac{dX^\mu}{d\tau} \frac{d}{dX^\mu}$; so we have:

$$\left(\frac{dX^\alpha}{d\tau} \right)_{;\mu} + \Gamma_{\mu\nu}^\alpha \frac{dX^\nu}{d\tau} = 0$$

This is the covariant derivative of four-velocity which is zero; so it vanishes in all coordinate systems.

Some properties of the Christoffel symbols: Christoffel symbols are not tensor (they are just functions of metric); they are symmetric under lower indices ($\Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha$).

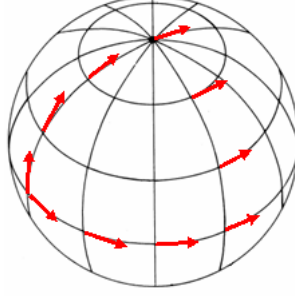


Figure 1.1: parallel transportation of a vector along a loop on a spherical space.

1.1.3 Space-Time Curvature

- Riemann's curvature tensor: After parallel transportation of a vector v^μ along a loop, its direction changes if the surface inside the loop, is curved; the changes of v^μ is dependent on the area inside loop, its curvature and the vector itself [2]:

$$\delta v^\mu \propto dX^\alpha dX^\beta v^\mu R_{\nu\alpha\beta}^\mu$$

$R_{\nu\alpha\beta}^\mu$ represents the curvature of the surface inside loop; it is called Riemann's curvature tensor and it can be written as:

$$R_{\nu\alpha\beta}^\mu = \Gamma_{\nu\beta,\alpha}^\mu - \Gamma_{\nu\alpha,\beta}^\mu + \Gamma_{\nu\beta}^\rho \Gamma_{\rho\alpha}^\mu - \Gamma_{\nu\alpha}^\rho \Gamma_{\rho\beta}^\mu \quad (1.10)$$

- Ricci curvature tensor: The contraction of Riemann's curvature tensor gives the Ricci curvature tensor as:

$$R_{\mu\nu} = R_{\mu\alpha\nu}^\alpha \quad (1.11)$$

or in more practical form:

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta \quad (1.12)$$

- Ricci scalar: The contraction of Ricci tensor; gives Ricci scalar:

$$R = R_\mu^\mu = g^{\mu\nu} R_{\mu\nu} \quad (1.13)$$

- Einstein's curvature tensor : Riemann's curvature tensors obey the relation below that is called Bianchi identity:

$$R_{\beta\lambda\mu;\nu}^\sigma + R_{\beta\nu\lambda;\mu}^\sigma + R_{\beta\mu\nu;\lambda}^\sigma = 0 \quad (1.14)$$

From the relation above, it can be proved that:

$$(R^\mu_\nu - \frac{1}{2}g^\mu_\nu R)_{;\mu} = 0 \quad (1.15)$$

So there is another tensor of rank-2 that its covariant derivative is zero; this tensor is called Einstein's curvature tensor [1,3]:

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}g^\mu_\nu R$$

or after multiplying it to metric tensor we have:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (1.16)$$

Because its covariant derivative is zero, the Einstein's curvature tensor is conserved under coordinate transformations.

1.1.4 Perfect Fluids

All the properties of a given fluid that change the shape of space-time are contributed in the energy-momentum tensor, $T_{\mu\nu}$. The energy-momentum tensor of a perfect fluid, that is a fluid without viscosity, is given by [1,2]:

$$T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu - pg_{\mu\nu} \quad (1.17)$$

where ρ is the fluid density and p is its stress, measured in the fluid's comoving rectangular coordinate system (rest frame). u_μ are the covariant components of the four-velocity in a arbitrary coordinate system. When measured in the rest frame, the four-velocity is $\hat{u}_\mu = (c, 0, 0, 0)$; so we have:

$$\hat{T}_{\mu\nu} = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{bmatrix} \quad (1.18)$$

1.1.5 Einstein Field Equation

The equation which relates the curvature of space-time to matter and its motion, is known as Einstein field equation; it can be written as [4]:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

or

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (1.19)$$

where G is the gravitational constant: $G \simeq 6.674 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$.

Because of the conservation of the Einstein's curvature tensor, $G_{\mu\nu}$, the energy-momentum tensor, $T_{\mu\nu}$, is conserved under coordinate transformations:

$$(T^\mu_\nu)_{;\mu} = 0$$

Which is the generalization of conservation laws for energy and momentum. Einstein's field equations can also be obtained from a variational principle. The corresponding action (or Einstein-Hilbert action) reads[5]:

$$S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g}(R - 2\Lambda) + \int d^4x \sqrt{-g}L_{mat}$$

Where Λ is the cosmological constant; L_{mat} is the lagrangian of matter fields; and $g = \det(g_{\mu\nu})$. The variation of the action with respect to the metric tensor $g_{\mu\nu}$, gives the more generalized form of Einstein's field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Where the definition of energy-momentum tensor is:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_{mat})}{\delta g_{\mu\nu}}$$

1.2 Cosmology

Cosmology is the study of the universe as a whole; it deals with the mechanism of the evolution of the universe and its components. According to the modern cosmology the universe begins with a Big Bang; and through the evolution of the universe, different fields split from each other and the components (fluids) of the universe evolve. While the universe expands, different structures (such as galaxies, black holes, stars and planets) emerge. In order to study of the universe we need to know its space-time geometry, and the fluids as its components and how they interact with each other and with space-time.

1.2.1 Cosmological Principle and the Robertson-Walker Metric

The most fundamental principle in order to study the universe is the cosmological principle, that is two symmetry assumptions:

- The universe is homogeneous in large scale: its properties are the same in any given point; so from a given observation point, the density is independent of the distance from the observer.
- The universe is isotropic: its properties are the same in all directions, so there is not any preferred observer in the universe and the observer sees the same density in all directions.

According to the cosmological principle the local curvature must be the same at all points of the universe; the line element for this universe can be written as [6]:

$$ds^2 = c^2 dt^2 - a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \quad (1.20)$$

which is called Robertson-Walker (RW) line element; where (r, θ, ϕ) are spatial comoving coordinates and, t is the cosmic time; as a function of cosmic time the factor $a(t)$ is called scale factor which represents the expansion of the universe; K is the curvature parameter and can be taken $-1, 0, +1$; when $K=1$ universe is closed without boundary and its spatial shape is a sphere with radius $a(t)$ at time t ; but when $K=0$ the universe is open and infinite with Euclidean space (flat infinite slices of space through the arrow of cosmic time); for $K=-1$, again the universe is open and infinite, but its spatial geometry is hyperbolic.

1.2.2 The Friedmann Equations

From the Robertson-Walker metric we can calculate Einstein's curvature tensor, $G_{\mu\nu}$. The metric tensor can be written as:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-Kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2 \sin^2 \theta \end{bmatrix} \quad (1.21)$$

From that, the non-zero components of the Ricci tensor are:

$$R_{00} = -3\frac{\ddot{a}}{a} \quad (1.22)$$

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2Kc^2}{c^2(1-Kr^2)} \quad (1.23)$$

$$R_{22} = \frac{r^2}{c^2}(a\ddot{a} + 2\dot{a}^2 + 2Kc^2) \quad (1.24)$$

$$R_{33} = \frac{r^2}{c^2}(a\ddot{a} + 2\dot{a}^2 + 2Kc^2) \sin^2 \theta \quad (1.25)$$

and the Ricci scalar is:

$$R = -\frac{6}{c^2} \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} \right] \quad (1.26)$$

In the comoving coordinate system defined by Robertson-Walker metric the fluid is at rest; so by substituting energy-momentum components, $T_{\mu\nu}$, from equation (1.18) and the components of $G_{\mu\nu}$ from relations above, in Einstein's field equations, equations (1.19), we find that:

$$3 \left(\frac{\dot{a}^2 + Kc^2}{a^2} \right) = 8\pi G\rho \quad (1.27)$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} = -8\pi G\frac{p}{c^2} \quad (1.28)$$

The equation (1.27) is the first Friedmann's equation; by substituting it in the equation (1.28) the second Friedmann's equation can be written as:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) \quad (1.29)$$

By definition, $H(t) \equiv \frac{\dot{a}}{a}$, is called Hubble parameter. $H(t)$ gives the expansion rate of the universe at a given cosmic time t , and its present value, $H_0 = H(t_0)$, is called Hubble constant. According to the measurements, the value of the Hubble constant is found as: $H_0 = (72 \pm 8) \text{ km s}^{-1} \text{ Mpc}^{-1}$

, where $Mpc = 3.09 \times 10^{19} \text{ km} = 3.26 \times 10^6 \text{ light-year}$. It is useful to introduce a dimensionless Hubble constant, defined as:

$$H_0 = 100h \text{ kms}^{-1}Mpc^{-1} \quad (1.30)$$

where $h \approx 0.72$.

1.2.3 Curvature of Universe

First we assume that the universe is flat, $K=0$. For this case the first Friedmann's equation, equation (1.27), will be written as:

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_c(t) \quad (1.31)$$

where $\rho_c(t)$ is the critical density of the universe at a given cosmic time; the critical density of the universe changes as the Hubble parameter evolves and it defines the curvature of the universe at any given time. If the total density (the whole density of different types of fluids) of the universe is bigger or smaller than its critical density at any given time, the universe is curved at that time; for the bigger total densities than the critical one the universe is closed without boundary (spherical space), and for the smaller total densities the universe is open and infinite (hyperbolic space). Today's value of the critical density is: $\rho_{c,0} = \rho_c(t_0)$. The another parameter called density parameter can be defined as:

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \quad (1.32)$$

where $\Omega_{tot}(t) = \frac{\rho_{tot}(t)}{\rho_c(t)} = 1$ represents a flat universe, $\Omega_{tot}(t) < 1$ an open universe, and $\Omega_{tot}(t) > 1$, a closed universe.

Observations shows that the density of the universe today, is very close to the today's critical density:

$$\rho_{tot,0} = \rho_{tot}(t_0) \approx \rho_{c,0} \quad (1.33)$$

or

$$\Omega_{tot,0} = \Omega_{tot}(t_0) \approx 1 \quad (1.34)$$

The measurments shows that today's value of the critical density of the universe is:

$$\rho_{c,0} \approx 1.879 \times 10^{-29} h^2 \text{ gcm}^{-3}$$

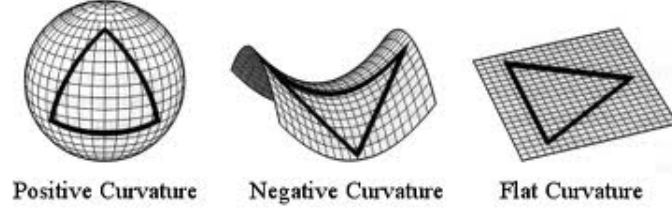


Figure 1.2: different possibilities of the curvature of a homogeneous and isotropic universe. The sum of the angles of a triangle on a flat Universe is 180 degrees, but in a closed universe the sum is greater than 180 and in an open universe the sum is less than 180.

1.2.4 Evolution of Fluids

The universe is an isolated system; so its heat content is constant and it expands adiabatically: $\delta Q = 0$

On the other hand all processes in the universe are reversible; so the second law of thermodynamics can be written as:

$$TdS = \delta Q$$

where S is the entropy and T is temprature.

So the universe is an isentropic system, this means that:

$$TdS = dE + pdV = 0$$

Where E is internal energy, p is pressure, and V is the volume. But the linear expansion of the homogeneous and isotropic universe demands that, $E = \rho c^2 V$ and $V \propto a^3$, and a and ρ are only functions of time ; so we have:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + \frac{p}{c^2}) \quad (1.35)$$

This is the continuity equation and from that we can find the evolution of energy density through the expansion of the universe.

Both the pressure and the energy density appear in this equation and the second Friedmann equation; so it is useful to define the equation of state which is a relation between these two parameters:

$$p = w\rho c^2 \quad (1.36)$$

w has different values for different fluids. When the fluid is pressureless (dust or bosonic and cold dark matter), $w=0$; and for radiation, $w = \frac{1}{3}$.

By substituting $P = w\rho c^2$ in continuity equation (1.35), we have:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\gamma\rho = 0 \quad (1.37)$$

where $\gamma = w + 1$, that is the another definition of equation of state. From the equation above we can find the evolution of the energy density through the expansion of the universe:

$$\rho = \rho_0 \left(\frac{a}{a_0} \right)^{-3\gamma} \quad (1.38)$$

In this equation ρ and a are the energy density and the scale factor, at a given time; and ρ_0 and a_0 are their today's values, respectively.

1.2.5 The Cosmic Redshift and Proper Distance

The wavelength of a light after emitting from a point in the universe, increases because of the expansion of the universe; so when it reaches to an observer, it has a longer value than it had at emission point. Light moves along a path defined by $ds^2 = 0$; by applying that to the Robertson-Walker line element and ignoring the contribution of peculiar velocity of the source point, we can find that:

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)} \quad (1.39)$$

where z is a parameter, common for measuring the cosmic redshift; $a(t_e)$ is the scale factor of universe when the light emitted and λ_e is the wavelength at emission point; $a(t_o)$ is the scale factor of the universe at the observation point and λ_o is the wavelength of light when observed. Today's value of scale factor is given as $a_0 = 1$, so the equation above will be simplified for today's observations (when $t_o = t_0$): $1 + z = \frac{\lambda_o}{\lambda_e} = \frac{1}{a(t_e)}$. The proper distance (or physical distance) is the spatial distance between any two events when the events are simultaneous (spatial geodesic). The proper distance of a emission point from an observer, is given by:

$$d_p(t_o) = \int_{t_e}^{t_o} c \frac{a(t_o)}{a(t)} dt \quad (1.40)$$

and its today's value (when $a(t_o) = a(t_0) = 1$), is called comoving proper distance (because it is measured at a fixed time, today).

1.2.6 Different Models of Universe

dependent on the equation of state, γ , and the curvature, the universe could expand in different ways; but γ itself represents the contribution of different fluid components in the energy density of the universe. In some epochs of the universe, only one fluid component dominates and the contribution of

the others are negligible; or there are epochs with two or more components; and also in order to study the formation and the evolution of the galaxies and the clusters of them, we can assume that they are sub-universes with curvature and components different from the whole universe. So theoretically we can talk about different models of universe with defined curvature, containing defined components, and see their evolutions and generalize the results to the real universe and its structures. Some simple models are as follows:

•Flat, dust dominated universe ($K = 0$ and $\gamma = 1$) ; this case is called Einstein-de Sitter model (EdS): The first Friedmann's equation, after substituting energy density (eq. 1.38) for this case, will be:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{0,m}\left(\frac{a}{a_0}\right)^{-3}$$

where $\rho_{0,m}$ is the today's value of energy density of dust. For an expanding universe ($\frac{\dot{a}}{a} > 0$), the equation above gives the evolution of scale factor a , as:

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \quad (1.41)$$

where $a=0$ at $t=0$; and t_0 is the present time which can be written as:

$$t_0 = \frac{2}{3H_0} \quad (1.42)$$

•Flat, radiation dominated universe ($K = 0$ and $\gamma = \frac{4}{3}$): For this case also, we can find that:

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}} \quad (1.43)$$

$$t_0 = \frac{1}{2H_0} \quad (1.44)$$

•Flat universe dominated by cosmological constant ($K = 0$ and $\gamma = 0$) called de Sitter model: Einstein introduced cosmological constant, Λ , in order to have a static universe; Λ represents the vacuum energy density, $\rho_\Lambda = \frac{\Lambda}{8\pi G} = \text{constant}$. So for a universe without radiation or matter the first Friedmann's equation can be written as:

$$H^2 = \frac{8\pi G}{3}\rho_\Lambda = \frac{8\pi G}{3}\rho_{\Lambda,0} = \frac{\Lambda}{3} = \text{constant} \quad (1.45)$$

For an expanding universe ($H(t) = \frac{\dot{a}}{a} > 0$), the relation above can be written as:

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} = \text{constant} \implies \frac{\dot{a}}{a} = H_0 \quad (1.46)$$

So we have:

$$a(t) = a_0 e^{H_0(t-t_0)} \quad (1.47)$$

Scale factor, $a(t)$, has finite value at $t=0$; so there is no singularity in this case (no Big Bang); and $\ddot{a}(t)$ is always positive, so the expansion of the universe is accelerating; the acceleration comes from negative pressure of vacuum energy density ($p_\Lambda = -\rho_\Lambda c^2$).

•Flat universe dominated by the positive cosmological constant and cold dark matter (the flat Λ CDM model): The first Friedmann's equation for this case is:

$$H^2 = \frac{8\pi G}{3} (\rho_{m,0} a^{-3} + \rho_{\Lambda,0}) = H_0^2 (\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}) \quad (1.48)$$

But, at any given time t , we have $\Omega_\Lambda = 1 - \Omega_m$ and the equation above could be written as:

$$H^2 = H_0^2 [\Omega_{m,0} a^{-3} + (1 - \Omega_{m,0})] \quad (1.49)$$

•Flat universe filled with matter and radiation:

$$H(t)^2 = \frac{8\pi G}{3} (\rho_m + \rho_r) = H_0^2 (\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4}) \quad (1.50)$$

At very early times radiation dominates the universe, but its energy density decreases as the universe expands, while the matter's energy density increases; at $a = a_{eq}$ both the fluids have the same energy density and after that time, matter dominates; so we have:

$$\rho_r(a_{eq}) = \rho_m(a_{eq}) \implies a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \quad (1.51)$$

•Standard cosmological model [7]: The universe containing radiation, matter and cosmological constant, with zero curvature is occasionally called standard cosmological model or the concordance cosmology. According to this model the early universe is dominated with radiation; then matter dominates after radiation-matter density equality era; then matter density decreases while the cosmological constant (or vacuum energy) increases; so the large amount of energy density of today's universe is cosmological constant:

$$\Omega_{\Lambda,0} \simeq 0.7$$

$$\Omega_{m,0} \simeq 0.3$$

$$\Omega_{r,0} \simeq 10^{-5}$$

1.2.7 Horizons

In the universe, according to the general theory of relativity, nothing can travel faster than the speed of light; and because of that the regions of the universe from which, we can receive information, is limited by event and particle horizons:

- Event horizon: we are limited by the event horizon in order to observe all the events in our future; the proper distance to the event horizon at time t , is given by:

$$d_p^{EH} = a(t) \int_t^\infty \frac{cdt'}{a(t')} \quad (1.52)$$

d_p^{EH} , has different values for different models of universe and its today's value is given by: $d_p^{EH}(t_0) = \int_{t_0}^\infty \frac{cdt'}{a(t')}$; so the events that happens today, at distances larger than $d_p^{EH}(t_0)$, will not be viewed in the future.

- Particle horizon: we cannot observe all the events happened in the past; the particle horizon defines our observable universe by the proper distance given by:

$$d_p^{PH} = a(t) \int_{t_{min}}^t \frac{cdt'}{a(t')} \quad (1.53)$$

Where t is the observation time and $t_{min} = 0$ is valid for cosmological models with Big Bang; similar to the event horizon, the proper distance to the particle horizon is dependent on the cosmological model of universe and its today's value is given by: $d_p^{PH}(t_0) = \int_{t_{min}}^{t_0} \frac{cdt'}{a(t')}$ which is the size of our observable universe today.

1.2.8 Accelerated Expansion of the Universe- Inflation

1:Inflation of early universe

According to the inflationary cosmology, universe experiences an accelerated, exponential expansion in its early stages, just after the Big Bang at about $t \sim 10^{-35}s$; this idea is introduced to solve the key problems of the ordinary Big Bang theory; the most important problems with the Big Bang theory without inflation are as follows[6,8,9]:

- Flatness problem: Observations shows that total energy density of the universe today, is close to its today's critical value $\rho_c(t_0) = \frac{3H_0^2}{8\pi G}$ (or total density parameter today $\Omega(t_0) \simeq 1$), which means that our universe is flat; according to the first Friedmann equation, any deviation of the density from its critical value at a given time, causes deviations in curvature of the

universe:

$$\Omega(t) - 1 = \frac{Kc^2}{a^2 H^2}$$

This deviation increases with time for a universe started with a Big Bang and filled with matter or radiation. So the energy density of the early universe must be very closer to its critical value, than it is today. Inflation (or exponential expansion) of the early universe, resolve this issue by driving energy density to be extremely close to its critical value at the end of the inflation, while the universe grows rapidly from planck scales to astronomical scales:

$$\frac{a(t_f)}{a(t_i)} = e^{H_i(t_f-t_i)} = e^N \quad (1.54)$$

Where t_i and t_f are the times when inflation starts and ends, respectively; $a(t_i)$ and $a(t_f)$ are scale factors of the universe at the beginning and ending of the inflation, respectively; and H_i is the hubble parameter at the beginning of the inflation which is constant during the inflation. N is called number of e-foldings which is a large number.

An important point here is that inflation theory resolves the issue of positive deviations of energy density; for negative deviations, dark matter is assumed to save the flat universe.

- Horizon problem: The universe is isotropic and homogeneous in large scale; it looks the same on opposite sides of the sky (opposite horizons); so there should have been communications between points with distances larger than particle horizon, in the past. Inflation of the early universe resolves this problem, too: The rapid exponential expansion of the universe from planck scales to the astronomical scales means that regions of the observable universe which are separated in the sky today, were much closer together before the inflation and they were in contact by light signals.

- Inflation theory resolves other problems too: It explains why we cannot observe any magnetic monopole in the sky; it explains the existence of galaxies and other structures and so the living beings in the universe, by producing small density fluctuations that can later in the history of the universe provide the seeds to cause matter to begin to clump together to form the galaxies and other observed structures.

2: Accelerated expansion of today's universe

Before 1990s most astronomers believed that expansion of the universe started by a Big Bang was decelerating and in the future it may turn into contraction; that was expected because of gravity force; but during 1990s the Hubble Space Telescope and ground-based telescopes allowed astronomers

to see almost the edges of the universe; they detected many supernova explosions ; they saw that the light coming from these stars had the same characteristics as the light coming from local supernovas, as they reached their maximum brightness and faded away; so there is no differences between distant supernovas and the local ones; the only difference is their brightness that could help astronomers to determine their distances; knowing how a supernova works one can determine its intrinsic brightness and comparing it with its appearance brightness it is possible to determine how far away it is (or how far away is the point on the sky that supernova was located there in the past); by doing so and measuring the redshifts of supernovas, astronomers found that our universe is expanding with a increasing rate ($\ddot{a} > 0$) [10], instead of decreasing due to the gravitational forces.

So we can conclud that universe has gone into different phases of expansion through its history: just after the Big Bang, when it was in planck scales, universe expands inflationary to astronomical scales; when inflation ends, radiation and then matter dominates the universe and its expansion slows down; but again it begins to speed up and today's expansion of universe is accelerating.

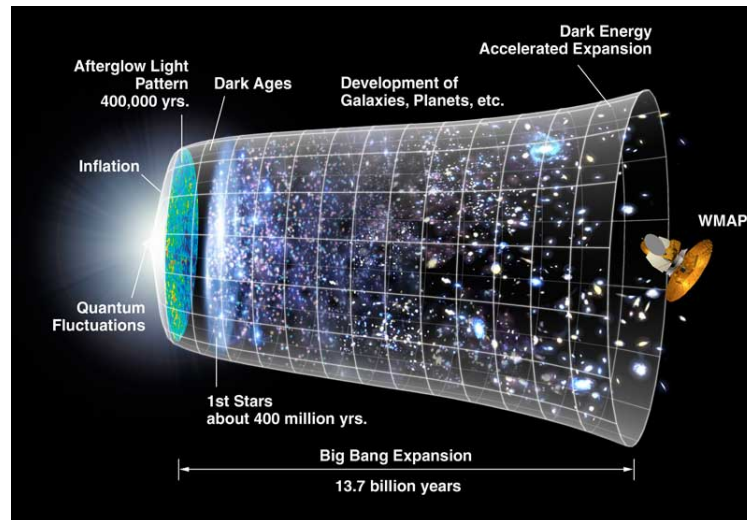


Figure 1.3: History of the universe from the Big Bang to the present day. Universe starts from quantum fluctuations of nothing (quantum vacuum); expands rapidly and exponentially (inflation); it slows down through the structure formation era; and now dark energy is speeding up the universe. (picture is from www.cosmotography.com)

1.2.9 Negative Pressure

We saw that our universe is flat, homogenous and isotropic; so it can be considered as flat FRW(Fridmann-Robertson-Walker) universe with the line element defined by:

$$ds^2 = dt^2 - a(t)^2 dx^2 \quad (1.55)$$

The universe contains different species (or fluids) defined by equation of state $P_i = (\gamma_i - 1)\rho_i$, where γ_i has different values for any kind of species; for instance $\gamma_i = 1$ for dust and $\gamma_i = \frac{4}{3}$ if the fluid is radiation. Densities of these species are constrained by Friedmann's first equation:

$$H^2 = \frac{\kappa^2}{3} \sum_i \rho_i \quad (1.56)$$

If universe expands adiabatically, the evolution of each component can be written as:

$$\dot{\rho}_i = -3H(\rho_i + p_i) \quad (1.57)$$

So the evolution of universe will be given by:

$$\dot{H} = -\frac{\kappa^2}{2} \sum_i (p_i + \rho_i) \quad (1.58)$$

The linear combinations above are valid if different species evolve independently.

The acceleration equation (second Friedmann's equation) of universe can be found from equations (1.56) and (1.58), that is:

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6} \sum_i (\rho_i + 3p_i) \quad (1.59)$$

If the expansion of the universe is accelerating ($\ddot{a} > 0$) at any time, the pressure of one of its components that is dominated at that time, must be minus and satisfy the relation below:

$$\rho_i + 3p_i < 0 \Rightarrow 1 + 3w_i < 0 \Rightarrow w_i < -\frac{1}{3} \quad \text{or} \quad \gamma_i < \frac{2}{3}$$

Matter is pressureless ($\gamma_m = 1$) and radiation with $\gamma_r = \frac{4}{3}$ has a positive pressure, so none of them could satisfy the relation above. So there is a need for another fluid with negative pressure to explain the inflation of the early universe or accelerating expansion of the today's universe that are proved by the observations. This fluid is called dark energy. Cosmological constant (or vacuum energy density), as discussed before, is the first and simplest model of dark energy; but there are another models of dark energy, called quintessential models. Unlike the cosmological constant which

has the same value everywhere in space for all the time, quintessence is a dynamical, evolving component of universe, with possibility to be spatially inhomogeneous[11].

Cosmological constant, if it comprises the dark energy, has not been fine-tuned to balance the matter; instead, the vacuum energy is overabundant, causing the expansion of the universe to accelerate; it is completely defined by one number, its magnitude[12]. The value of energy density of vacuum, based on the result of different theories, is $10^{50} - 10^{120}$ times larger than the magnitude allowed by cosmology[13].

There are different models of quintessential approach to negative pressure (dark energy and inflation), such as: scalar field[14,15], three-form[16], tachyon[17], non-metric quintessential model[18], ...

There are also, alternatives to early inflation of universe. For instance varying speed of light scenario that assumes the speed of light in the very early universe was much larger than it is today[19]; or the cyclic theory which assumes that the big bang is not the beginning of space and time[20].

In the next chapter the scalar field will be discussed and the chapter three will be analysis of equations derived from non-metric quintessential model of dark energy (or non-metric chaotic inflation).

Chapter 2

Cosmological Scaling Solutions

2.1 Scalar Field

Homogeneous scalar field $\Phi(t)$ which is the simplest form of matter with a negative pressure can be defined as [14,21,22]:

$$\rho_\Phi(t) = \frac{1}{2}\dot{\Phi}^2 + V(\Phi) \quad (2.1)$$

$$p_\Phi(t) = \frac{1}{2}\dot{\Phi}^2 - V(\Phi) \quad (2.2)$$

In this model $\Phi(t)$ behaves as a perfect fluid, and $V(\Phi)$ is its potential. Based on adiabatic expansion of universe, the evolution of scalar field density can be written as:

$$\dot{\rho}_\Phi(t) = -3H[\rho_\Phi(t) + p_\Phi(t)] \quad (2.3)$$

But ρ_Φ and p_Φ can be substituted from equations (2.1) and (2.2) :

$$\begin{aligned} \left(\frac{1}{2}\dot{\Phi}^2 + V(\Phi)\right)' &= -3H\dot{\Phi}^2 \\ \Rightarrow \ddot{\Phi}\dot{\Phi} + \dot{V} &= -3H\dot{\Phi}^2 \end{aligned}$$

and using the chain rule, \dot{V} can be substituted as: $\dot{V} = \frac{dV(\Phi)}{dt} = \frac{dV(\Phi)}{d\Phi}\dot{\Phi}$, so we have:

$$\ddot{\Phi} + \frac{dV}{d\Phi} + 3H\dot{\Phi} = 0 \quad (2.4)$$

This equation is Klein-Gordon equation, that is the equation of motion of scalar field.

Now for a spatially flat Friedmann-Robertson-Walker (FRW) universe, containing two species, scalar field fluid, and a background fluid with equation

of state defined as $p_{bg} = (\gamma - 1)\rho_{bg}$, the second Friedmann equation (or evolution of space) can be written as:

$$\dot{H} = -\frac{\kappa^2}{2} \sum_i (\rho_i + p_i) = -\frac{\kappa^2}{2} (\gamma\rho_{bg} + \dot{\Phi}^2) \quad (2.5)$$

Through the evolution, these species are constrained by Friedmann's first equation:

$$H^2 = \frac{\kappa^2}{3} (\rho_{bg} + \frac{1}{2}\dot{\Phi}^2 + V) \quad (2.6)$$

Using dimensionless variables, x and y , defined as:

$$x = \sqrt{\frac{\kappa^2 \dot{\Phi}^2}{6H^2}}, \quad y = \sqrt{\frac{\kappa^2 V}{3H^2}} \quad (2.7)$$

the equation (2.6) can be written as:

$$x^2 + y^2 + \frac{\kappa^2 \rho_{bc}}{3H^2} = 1 \quad (2.8)$$

Where the density parameters of scalar field and background fluid for a flat universe, are:

$$\Omega_{\Phi} = x^2 + y^2 = \frac{\kappa^2 \rho_{\Phi}}{3H^2}, \quad \Omega_{bg} = \frac{\kappa^2 \rho_{bg}}{3H^2} \quad (2.9)$$

And the effective equation of state for a scalar field, similar to the background field, can be found as:

$$\gamma_{\Phi} = \frac{\rho_{\Phi} + p_{\Phi}}{\rho_{\Phi}} = \frac{\dot{\Phi}^2}{V + \frac{1}{2}\dot{\Phi}^2} = \frac{2x^2}{x^2 + y^2} \quad (2.10)$$

x and y describe the kinetic and potential energy of the scalar field. The dynamics of these variables, gives the evolution of the scalar field; it is convenient to use their derivatives with respect to logarithm of scale factor, $N = \ln(a)$, instead of time t :

$$' = \frac{d}{dN} = \frac{da}{dN} \frac{dt}{da} \frac{d}{dt} = \frac{a}{\dot{a}} \frac{d}{dt} = \frac{1}{H} \frac{d}{dt}$$

So the derivatives of x and y , can be written as:

$$x' = \frac{1}{H} \dot{x} = \frac{1}{H} \frac{\kappa}{\sqrt{6}} \frac{d}{dt} \left(\frac{\dot{\Phi}}{H} \right) = \frac{1}{H} \frac{\kappa}{\sqrt{6}} \left(\frac{\ddot{\Phi}}{H} - \dot{\Phi} \frac{\dot{H}}{H^2} \right) \quad (2.11)$$

$$y' = \frac{1}{H} \dot{y} = \frac{1}{H} \frac{\kappa}{\sqrt{3}} \frac{d}{dt} \left(\frac{\sqrt{V}}{H} \right) = \frac{1}{H} \frac{\kappa}{\sqrt{3}} \left(\frac{1}{2} \frac{\dot{V}}{\sqrt{V}H} - \sqrt{V} \frac{\dot{H}}{H^2} \right) \quad (2.12)$$

In equation (2.11), $\ddot{\Phi}$ can be substituted from equation(2.4) ,or Klein-Gordon equation; and $\frac{\dot{H}}{H^2}$ can be written as:

$$\begin{aligned}\frac{\dot{H}}{H^2} &= \frac{-\frac{\kappa^2}{2}(\gamma\rho_{bg} + \dot{\Phi}^2)}{H^2} = -\frac{3}{2}\gamma\Omega_{bg} - 3x^2 \implies \\ \frac{\dot{H}}{H^2} &= -\frac{3}{2}\gamma(1 - x^2 - y^2) - 3x^2\end{aligned}\quad (2.13)$$

If the potential is the exponential potential, defined as [23]:

$$V = V_0 e^{-\lambda\kappa\Phi} \quad (2.14)$$

where λ is a constant and we have:

$$\dot{V} = \frac{dV}{d\Phi} \frac{d\Phi}{dt} = -\lambda\kappa V \dot{\Phi}$$

So, by substituting $\ddot{\Phi}$, V , \dot{V} and $\frac{\dot{H}}{H^2}$, and using equations (2.7), equations (2.11) and (2.12) will be written as:

$$x' = -3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x[2x^2 + \gamma(1 - x^2 - y^2)] \quad (2.15)$$

$$y' = -\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y[2x^2 + \gamma(1 - x^2 - y^2)] \quad (2.16)$$

Equations (2.15) and (2.16), form a nonlinear autonomous system on a phase plane; because of nonlinearity, this system does not have exact solution; but it can be analyzed qualitatively, and can be solved numerically.

2.2 Analysing Autonomous System

From constraint equation (2.8) for flat universe, it is obvious that:

$$0 \leq x^2 + y^2 \leq 1$$

So x and y evolve within a disc of unit radius:

$$\begin{aligned}-1 &\leq x \leq 1 \\ -1 &\leq y \leq 1\end{aligned}$$

Under the reflection $(x, y) \rightarrow (x, -y)$, the (x', y') reflects to $(x', -y')$; so the autonomous system of (x', y') is symmetric under this reflection and we study only the upper half-plane of the disc; the lower half-plane is the same as upper one.

2.2.1 Equilibrium Solutions

Critical points (or equilibrium solutions, (x_c, y_c) , are the solutions of the autonomous system when $x' = y' = 0$; so at the critical points, equations (2.15) and (2.16), will be written as:

$$-3x_c + \lambda\sqrt{\frac{3}{2}}y_c^2 + \frac{3}{2}x_c[2x_c^2 + \gamma(1 - x_c^2 - y_c^2)] = 0 \quad (2.17)$$

$$-\lambda\sqrt{\frac{3}{2}}x_c y_c + \frac{3}{2}y_c[2x_c^2 + \gamma(1 - x_c^2 - y_c^2)] = 0 \quad (2.18)$$

Two cases can be considered in order to find (x_c, y_c) :

1- $y_c \neq 0$: in this case the equation (2.18) can be divided by y_c , that gives:

$$y_c^2 = \left(\frac{2}{\gamma} - 1\right)x_c^2 - \frac{\lambda}{\gamma}\sqrt{\frac{2}{3}}x_c + 1 \quad (2.19)$$

by substituting the answer above in equation (2.17), we can find that:

$$\frac{\lambda}{\gamma}\sqrt{6}x_c^2 - \left(3 + \frac{\lambda^2}{\gamma}\right)x_c + \lambda\sqrt{\frac{3}{2}} = 0$$

from this and relation (2.19), four critical points can be found as:

$$\left(\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}, \pm\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}\sqrt{\frac{2}{\gamma} - 1}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{6}}\lambda, \pm\sqrt{1 - \frac{\lambda^2}{6}}\right)$$

The symmetry mentioned before, can be seen here; but the positive answers of y_c are enough for our discussion.

•At point $\left(\frac{1}{\sqrt{6}}\lambda, \pm\sqrt{1 - \frac{\lambda^2}{6}}\right)$, density parameter of scalar field $\Omega_\Phi = 1$; so this point is scalar field dominated; the effective equation of state of the scalar field at this point is: $\gamma_\Phi = \frac{1}{3}\lambda^2$.

•At point $\left(\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}, \pm\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}\sqrt{\frac{2}{\gamma} - 1}\right)$, where $\Omega_\Phi = 3\frac{\gamma}{\lambda^2}$, none of the fluids entirely dominates and we have a scaling solution; the effective equation of state of the scalar field at this point is: $\gamma_\Phi = \gamma$.

2- $y_c = 0$: in this case the equation (2.17), will be written as:

$$-3x_c + \frac{3}{2}x_c[2x_c^2 + \gamma(1 - x_c^2)] = 0$$

from this relation three other critical points can be found as:

$$\begin{cases} (-1, 0) & , \text{ if } \gamma \neq 2 \\ (0, 0) \\ (+1, 0) & , \text{ if } \gamma \neq 2 \end{cases}$$

At points $(\pm 1, 0)$, where $\Omega_\Phi = 1$, the scalar field dominates, but only its kinetic energy; and $\gamma_\Phi = 2$ at these points. At point $(0, 0)$, $\Omega_\Phi = 0$, so background fluid dominates ($\Omega_{bg} = 1$); and the effective equation of state of scalar field is undefined.

2.2.2 Validity of Critical Points

As mentioned before dimensionless variables, x and y , are always within a disc of radius one; so the critical points, x_c and y_c , must comply these conditions: $x_c^2 \leq 1$, $y_c^2 \leq 1$ and $x_c^2 + y_c^2 \leq 1$. Three critical points on y -axis, $\{(-1,0), (0,0), (1,0)\}$, are consistent with these conditions; but $(\pm 1,0)$ are valid when $\gamma \neq 2$. For the two (or four) other points, the conditions above require that:

$$1: \left(\sqrt{\frac{3}{2}} \frac{\gamma}{\lambda}, \pm \sqrt{\frac{3}{2}} \frac{\gamma}{\lambda} \sqrt{\frac{2}{\gamma} - 1} \right) \Rightarrow \begin{cases} \frac{3}{2} \frac{\gamma^2}{\lambda^2} \leq 1 \\ \frac{3}{2} \frac{\gamma^2}{\lambda^2} \left(\frac{2}{\gamma} - 1 \right) \leq 1 \\ 3 \frac{\gamma}{\lambda^2} \leq 1 \end{cases} \Rightarrow \begin{cases} \lambda^2 \geq \frac{3}{2} \gamma^2 \\ \lambda^2 \geq 3\gamma - \frac{3}{2} \gamma^2 \\ \lambda^2 \geq 3\gamma \end{cases}$$

$$\Rightarrow \lambda^2 \geq 3\gamma \text{ and } \gamma \leq 2$$

$$2: \left(\frac{\lambda}{\sqrt{6}}, \pm \sqrt{1 - \frac{\lambda^2}{6}} \right) \Rightarrow \begin{cases} \frac{\lambda^2}{6} \leq 1 \\ 1 - \frac{\lambda^2}{6} \leq 1 \\ 1 \leq 1 \end{cases} \Rightarrow \begin{cases} \lambda^2 \leq 6 \\ \lambda^2 \geq 0 \end{cases} \Rightarrow \lambda^2 \leq 6$$

when $\lambda^2 = 6$, this point is the same as $(1,0)$; so $\lambda^2 < 6$.

2.2.3 Linear Stability of Critical Points

A critical point (or equilibrium condition) is stable if after a small perturbation, system stays in the vicinity of this point; so in order to study a critical point, the points around it must be investigated. To do that, equations (2.15) and (2.16), is better to be written as:

$$\begin{cases} x'(N) = f(x, y) \\ y'(N) = g(x, y) \end{cases} \quad (2.20)$$

Near a critical point, (x_c, y_c) , the dynamics of this nonlinear system can be approximated by dynamics of a linear system (if exists), that is the first-order terms of Taylor expansion around (x_c, y_c) [24]:

$$\begin{cases} x'(N) = f(x, y) \approx f(x_c, y_c) + \frac{\partial f}{\partial x}|_{(x_c, y_c)}(x - x_c) + \frac{\partial f}{\partial y}|_{(x_c, y_c)}(y - y_c) \\ y'(N) = g(x, y) \approx g(x_c, y_c) + \frac{\partial g}{\partial x}|_{(x_c, y_c)}(x - x_c) + \frac{\partial g}{\partial y}|_{(x_c, y_c)}(y - y_c) \end{cases} \quad (2.21)$$

where $f(x_c, y_c) = g(x_c, y_c) = 0$; and the relations above can be represented as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = J(x_c, y_c) \begin{bmatrix} x - x_c \\ y - y_c \end{bmatrix}$$

$J(x_c, y_c)$ is the Jacobian (or coefficient) matrix of the linear system at critical point (x_c, y_c) :

$$J(x_c, y_c) = \begin{bmatrix} -3 + 9x_c^2 + \frac{3}{2}\gamma - \frac{9}{2}\gamma x_c^2 - \frac{3}{2}\gamma y_c^2 & \sqrt{6}\lambda y_c - 3\gamma x_c y_c \\ -\lambda\sqrt{\frac{3}{2}}y_c + 6x_c y_c - 3\gamma x_c y_c & -\lambda\sqrt{\frac{3}{2}}x_c + 3x_c^2 + \frac{3}{2}\gamma - \frac{3}{2}\gamma x_c^2 - \frac{9}{2}\gamma y_c^2 \end{bmatrix} \quad (2.22)$$

Poincare-Lyapunov Theorem: If the eigenvalues of the Jacobian matrix, found for linear system, are not zero or pure imaginary numbers, trajectories of the nonlinear system around a critical point behave the same way as the trajectories of the associated linear system.

If the eigen values of a Jacobian matrix defined for a fixed point are not zero or pure imaginary numbers, that point is called hyperbolic point. Non-hyperbolic critical points are not linearizable.

There are three different types of critical hyperbolic points [25]:

(I) if the eigenvalues are negative or their real parts are negative if they are complex, then the critical point is an attractor (or sink).

(II) if the eigenvalues are positive or complex with positive real part, then the critical point is a source (or repeller), that is unstable.

(III) if the eigenvalues are real but with different sign, then the critical point is a saddle point.

Stability of five critical points on the upper half of the phase plane:

• $(\pm 1, 0)$: valid for $\gamma \neq 2$, these two points describe domination of kintetic energy of scalar field; and the Jacobian at these points can be written as:

$$J(\pm 1, 0) = \begin{bmatrix} 6 - 3\gamma & 0 \\ 0 & 3 \mp \lambda\sqrt{\frac{3}{2}} \end{bmatrix}$$

that is diagonal, and from this Jacobian , we can see that $\gamma \neq 2$, otherwise the linearization crashes. linearization craches also when $\lambda = \sqrt{6}$ and $\lambda = -\sqrt{6}$, for point $(1,0)$ and $(-1,0)$, respectively.

(I) When $\gamma < 2$, one of the eignvalues is positive; then $(1,0)$ for $\lambda < \sqrt{6}$, is a unstable point (repeller), and for $\lambda > \sqrt{6}$ this point is a saddle; but the point $(-1,0)$ is unstable if $\lambda > -\sqrt{6}$ and saddle if $\lambda < -\sqrt{6}$.

(II) When $\gamma > 2$; then $(1,0)$ for $\lambda < \sqrt{6}$ is a saddle point, and for $\lambda > \sqrt{6}$, this point is an attractor; but $(-1,0)$ is a saddle point when $\lambda > -\sqrt{6}$, and

an attractor when $\lambda < -\sqrt{6}$.

•(0,0): at this point background fluid dominates; and we have:

$$J(0,0) = \begin{bmatrix} -3 + \frac{3}{2}\gamma & 0 \\ 0 & \frac{3}{2}\gamma \end{bmatrix}$$

linearization at this point crashes when γ is zero or two; so when $0 < \gamma < 2$ this point is saddle; but for $\gamma > 2$ this point is a source.

•($\sqrt{\frac{3}{2}\frac{\gamma}{\lambda}}$, $\sqrt{\frac{3}{2}\frac{\gamma}{\lambda}}\sqrt{\frac{2}{\gamma}-1}$): At this point where we have scaling solution, Jacobian can be written as:

$$J\left(\sqrt{\frac{3}{2}\frac{\gamma}{\lambda}}, \sqrt{\frac{3}{2}\frac{\gamma}{\lambda}}\sqrt{\frac{2}{\gamma}-1}\right) = \begin{bmatrix} -\frac{9}{2}\frac{\gamma^3}{\lambda^2} + 9\frac{\gamma^2}{\lambda^2} + \frac{3}{2}\gamma - 3 & 3\gamma\sqrt{\frac{2}{\gamma}-1}\left(1 - \frac{3}{2}\frac{\gamma^2}{\lambda^2}\right) \\ 3\gamma\sqrt{\frac{2}{\gamma}-1}\left(-\frac{1}{2} - \frac{3}{2}\frac{\gamma^2}{\lambda^2} + 3\frac{\gamma}{\lambda^2}\right) & \frac{9}{2}\frac{\gamma^3}{\lambda^2} - 9\frac{\gamma^2}{\lambda^2} \end{bmatrix}$$

This matrix is not diagonal; after diagonalization, its eigenvalues can be found as:

$$m_{\pm} = -\frac{3(2-\gamma)}{4} \left[1 \pm \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2-\gamma)}} \right]$$

For this point also, $\gamma \neq 2$ and $\lambda^2 \neq 3\gamma$ unless the linearization crashes; from these conditions and according to the validity conditions, we have $\gamma < 2$ and $\lambda^2 > 3\gamma$ for this point; the point is stable if the term inside the bracket is positive or its real part is positive; in fact it can not be real negative; so:

(I) if $\frac{8\gamma(\lambda^2-3\gamma)}{\lambda^2(2-\gamma)} < 1 \implies \lambda^2 < \frac{24\gamma^2}{9\gamma-2}$, the point is an attractor;

(II) if $\lambda^2 > \frac{24\gamma^2}{9\gamma-2}$, both the eigenvalues are complex with minus real part; so the critical point is a stable spiral (that is attractor).

•($\frac{\lambda}{\sqrt{6}}$, $\sqrt{1 - \frac{\lambda^2}{6}}$): This point that is where the scalar field dominates, is defined for $\lambda^2 < 6$ and its Jacobian is:

$$J\left(\frac{\lambda}{\sqrt{6}}, \sqrt{1 - \frac{\lambda^2}{6}}\right) = \begin{bmatrix} -3 + \frac{1}{2}\lambda^2(3-\gamma) & \lambda\sqrt{1 - \frac{\lambda^2}{6}}(\sqrt{6} - \sqrt{\frac{3}{2}\gamma}) \\ \lambda\sqrt{\frac{3}{2}(1 - \frac{\lambda^2}{6})}(1-\gamma) & -3\gamma + \frac{1}{2}\gamma\lambda^2 \end{bmatrix}$$

and after diagonalization, its eigen values are:

$$m_{\pm} = \frac{-(3 + 3\gamma - \frac{3}{2}\lambda^2) \pm \sqrt{(3 + 3\gamma - \frac{3}{2}\lambda^2)^2 - 2(\lambda^2 - 6)(\lambda^2 - 3\gamma)}}{2}$$

Different cases for this critical point, are as follows:

a) saddle point , if $(\lambda^2 - 6)(\lambda^2 - 3\gamma) < 0$; but $(\lambda^2 - 6) < 0$, so $(\lambda^2 - 3\gamma) > 0$ or $\lambda^2 > 3\gamma$.

b) if $\lambda^2 < 3\gamma$ and $(3 + 3\gamma - \frac{3}{2}\lambda^2)^2 > 2(\lambda^2 - 6)(\lambda^2 - 3\gamma)$ then:

-if $\lambda^2 < 2(1 + \gamma)$; the point is an attractor.

-if $\lambda^2 > 2(1 + \gamma)$; the point is repeller.

c) if $\lambda^2 < 3\gamma$ and $(3 + 3\gamma - \frac{3}{2}\lambda^2)^2 < 2(\lambda^2 - 6)(\lambda^2 - 3\gamma)$ then:

-if $\lambda^2 < 2(1 + \gamma)$; the point is an attractor spiral.

-if $\lambda^2 > 2(1 + \gamma)$; the point is repeller spiral.

d)if $\lambda^2 = 3\gamma$, or $\lambda^2 = 6$; linearization crashes.

2.2.4 Phase Planes

Figures 2.1 through 2.6 shows the phase planes for some different values of γ and λ . There are four or five critical points in these planes where $x' = 0$ and $y' = 0$; two of them, $(x = \pm 1, y = 0)$, represent domination of kinetic energy of the scalar field with $\gamma_\Phi = 2$ and one of them, $(x=0, y=0)$, with undefined γ_Φ is the fixed point where background fluid dominates ($\Omega_\Phi = 0$ and $\Omega_{bg} = 1$).

The two other equilibrium conditions are either scalar field dominated points ($\Omega_\Phi = 1$) or scaling solutions ($\Omega_\Phi = 3\frac{\gamma}{\lambda^2}$):

-Scalar field dominated critical points appears when $\lambda^2 < 6$ (according to validity conditions); we can see this point at figures (2.1), (2.2), (2.4) and (2.5).

-At critical points where $\lambda^2 \geq 3\gamma$ and $\gamma \leq 2$, we have scaling solution (equilibrium contributed by scalar field and background fluid); we can see this point at figures (2.2), (2.3), (2.5) and (2.6) ; but at figure (2.5) we have $\lambda^2 = 3\gamma \implies \Omega_\Phi = 3\frac{\gamma}{\lambda^2} = 1$, so this point becomes scalar field dominated.

Stability of these critical points have been investigated according to the conditions given in the previous section; except for the non-hyperbolic point at figure (2.5).

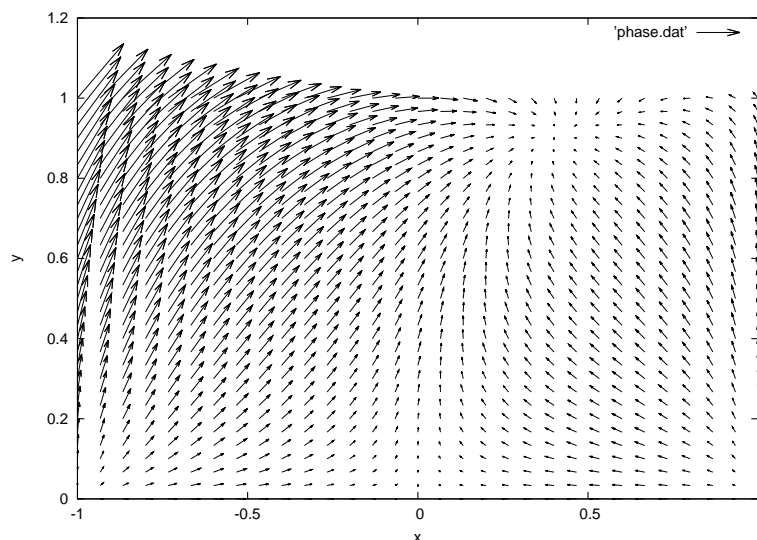


Figure 2.1: The phase plane for $\gamma = 1$, $\lambda = 1$: There are source points at $(1,0)$ and $(-1,0)$; saddle at $(0,0)$ and attractor (or sink) at $(\frac{1}{\sqrt{6}}, \frac{\sqrt{5}}{6})$.

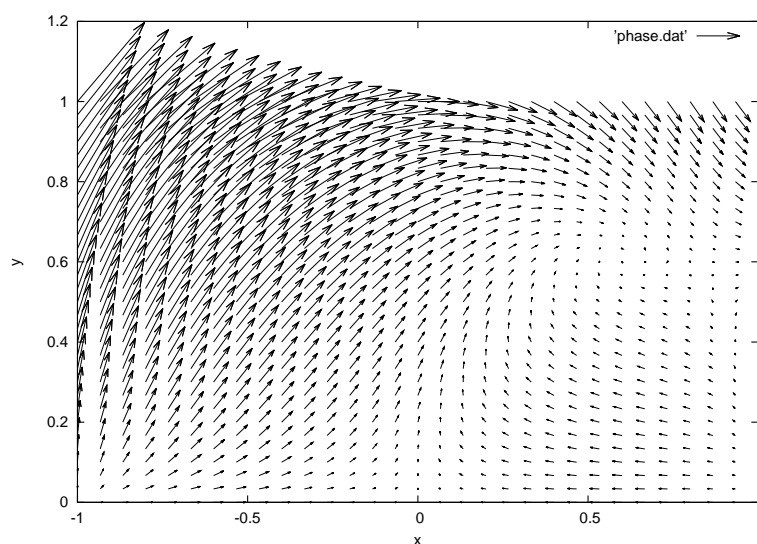


Figure 2.2: The phase plane for $\gamma = 1$, $\lambda = 2$: There are source points at $(1,0)$ and $(-1,0)$; saddle at $(0,0)$ and attractor spiral at $(\frac{\sqrt{3}}{8}, \frac{\sqrt{3}}{8})$ and a saddle point at $(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}})$.

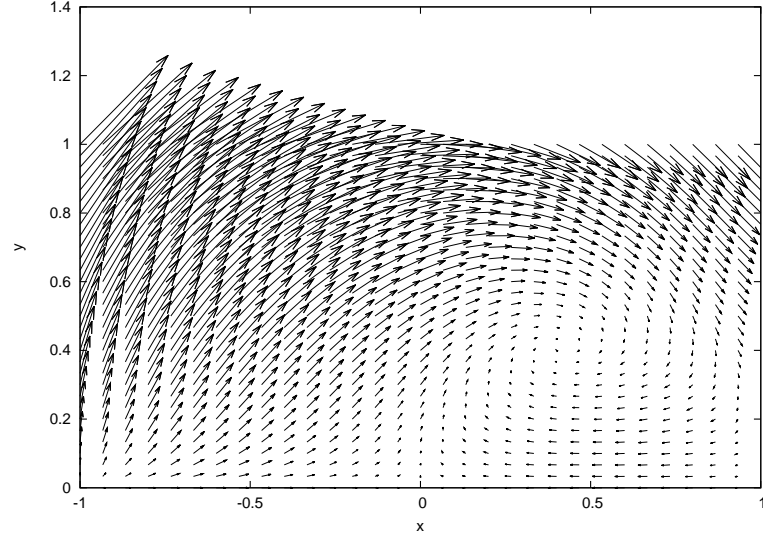


Figure 2.3: The phase plane for $\gamma = 1$, $\lambda = 3$: There are saddle points at $(1,0)$ and $(0,0)$; repeller at $(-1,0)$ and attractor spiral at $(\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{6}})$.

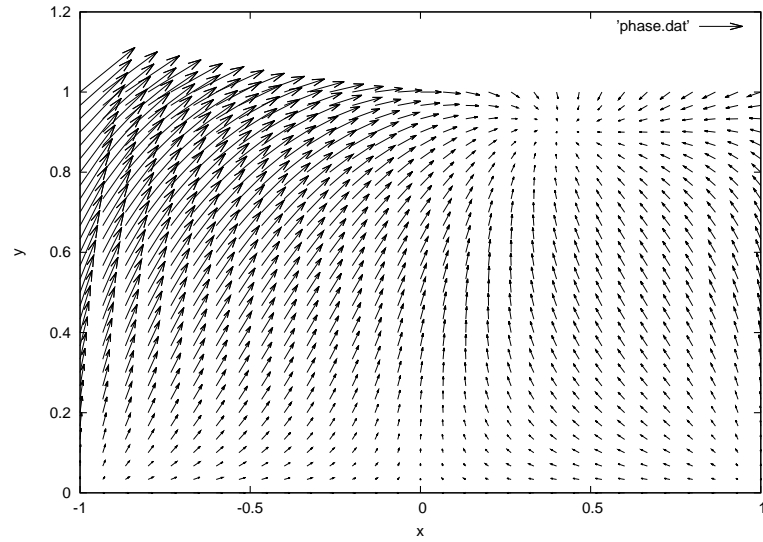


Figure 2.4: The phase plane for $\gamma = \frac{4}{3}$, $\lambda = 1$: There are source points at $(1,0)$ and $(-1,0)$; saddle at $(0,0)$ and attractor at $(\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}})$.

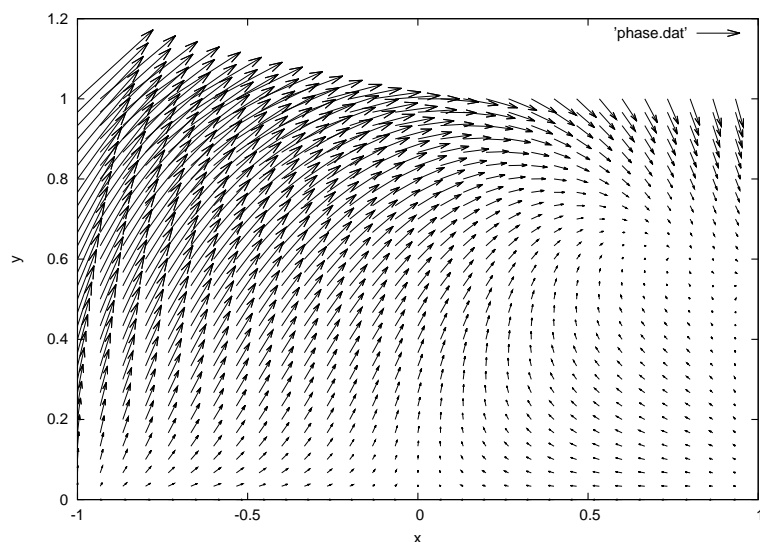


Figure 2.5: The phase plane for $\gamma = \frac{4}{3}$, $\lambda = 2$: There are source points at $(1,0)$ and $(-1,0)$; saddle at $(0,0)$ and one non-hyperbolic point at $(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}})$.

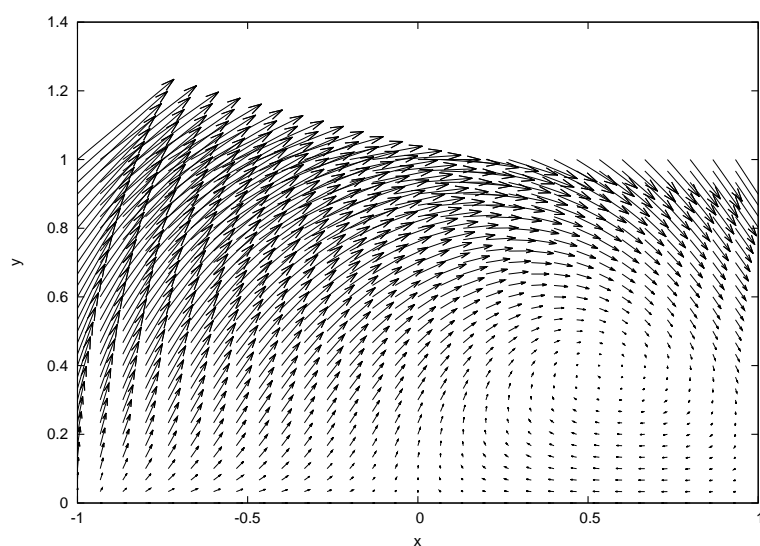


Figure 2.6: The phase plane for $\gamma = \frac{4}{3}$, $\lambda = 3$: There is repeller at $(-1,0)$; saddles at $(1,0)$ and $(0,0)$; attractor spiral at $(\sqrt{\frac{3}{8}}, \sqrt{\frac{3}{8}})$.

2.3 Evolution of Universe filled with Scalar Field and Background Fluids

Although phase plans give some information about equilibrium conditions of our universe, but in order to get better understanding of evolution of universe, the equations of autonomous system must be solved; but first the equations of the evolution of background fluids must be added to the system. The Friedmann's constraining equation for a universe containing scalar field and radiation with equation of state $\gamma = \frac{4}{3}$, and dust (included cold dark matter and baryonic matter) with equation of state $\gamma = 1$, can be written as:

$$x^2 + y^2 + r^2 + m^2 = 1 \quad (2.23)$$

Where r^2 and m^2 are dimensionless variables (density parameters) of radiation and matter respectively, defined as:

$$r^2 = \frac{\kappa^2 \rho_r}{3H^2}, \quad m^2 = \frac{\kappa^2 \rho_m}{3H^2} \quad (2.24)$$

In equations (2.15) and (2.16) too, γ can be canceled as follows:

$$\gamma(1 - x^2 - y^2) = \frac{\kappa^2 \rho_m}{3H^2} + \frac{4}{3} \frac{\kappa^2 \rho_r}{3H^2} = m^2 + \frac{4}{3} r^2$$

and by using constraint equation again, gives:

$$\gamma(1 - x^2 - y^2) = 1 - x^2 - y^2 + \frac{1}{3} r^2$$

Now one can write equations (2.15) and (2.16) as:

$$x' = -3x + \lambda \sqrt{\frac{3}{2}} y^2 + \frac{3}{2} x [1 + x^2 - y^2 + \frac{1}{3} r^2] \quad (2.25)$$

$$y' = -\lambda \sqrt{\frac{3}{2}} xy + \frac{3}{2} y [1 + x^2 - y^2 + \frac{1}{3} r^2] \quad (2.26)$$

The evolution of radiation, r' , and matter m' , can be calculated as:

$$r' = \frac{1}{H} \dot{r} = \frac{\kappa}{\sqrt{3}H} \frac{d}{dt} \left(\frac{\sqrt{\rho_r}}{H} \right) = \frac{\kappa}{\sqrt{3}H} \left(\frac{1}{2} \frac{\dot{\rho}_r}{\sqrt{\rho_r}H} - \sqrt{\rho_r} \frac{\dot{H}}{H^2} \right) \quad (2.27)$$

$$m' = \frac{1}{H} \dot{m} = \frac{\kappa}{\sqrt{3}H} \frac{d}{dt} \left(\frac{\sqrt{\rho_m}}{H} \right) = \frac{\kappa}{\sqrt{3}H} \left(\frac{1}{2} \frac{\dot{\rho}_m}{\sqrt{\rho_m}H} - \sqrt{\rho_m} \frac{\dot{H}}{H^2} \right) \quad (2.28)$$

But universe expands adiabatically, so:

$$\dot{\rho}_r = -3H(\rho_r + p_r) = -4H\rho_r$$

and

$$\dot{\rho}_m = -3H(\rho_m + p_m) = -3H\rho_m$$

And similar to equation (2.13), we have:

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}m^2 - 2r^2 - 3x^2 \quad (2.29)$$

So the relations (2.27) and (2.28) will be written as:

$$r' = r(-2 + \frac{3}{2}m^2 + 2r^2 + 3x^2) \quad (2.30)$$

$$m' = m(-\frac{3}{2} + \frac{3}{2}m^2 + 2r^2 + 3x^2) \quad (2.31)$$

four nonlinear equations (2.25), (2.26), (2.30) and (2.31), make an autonomous system again; it can be solved numerically, which gives the evolution of kinetic and potential energy of scalar field and the evolution of radiation and matter, while universe expands. As a initial conditions, we can assume that radiation dominates early universe and the other fluids are negligible compared to the radiation. The following pictures show the evolution of density parameters and scalar field, for some different values of λ .

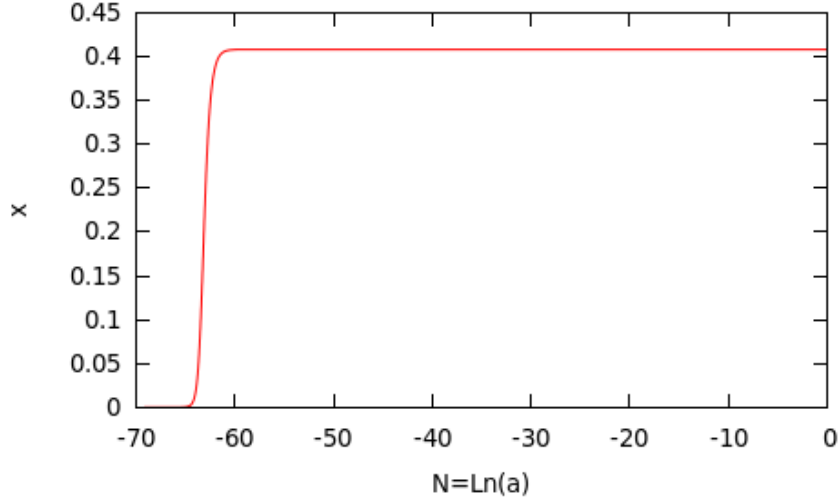


Figure 2.7: Evolution of kinetic energy of scalar field, x , when $\lambda = 1$.

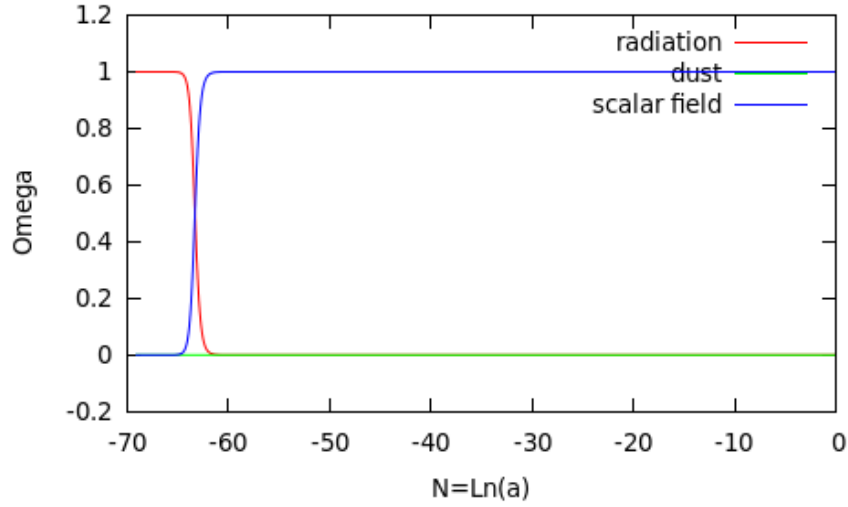


Figure 2.8: Evolution of dynamical variables, when $\lambda = 1$. Today's value of Hubble parameter for this case is: $H_0 \simeq 4.6 \times 10^{21} S^{-1}$, that is very large compared to the its observed value; and matter never appears in this universe.

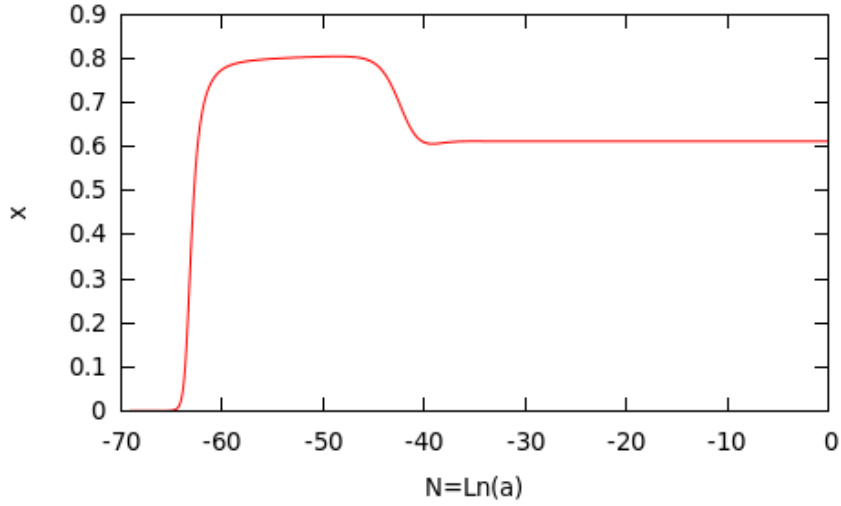


Figure 2.9: Evolution of kinetic energy of scalar field, x , when $\lambda = 2$.

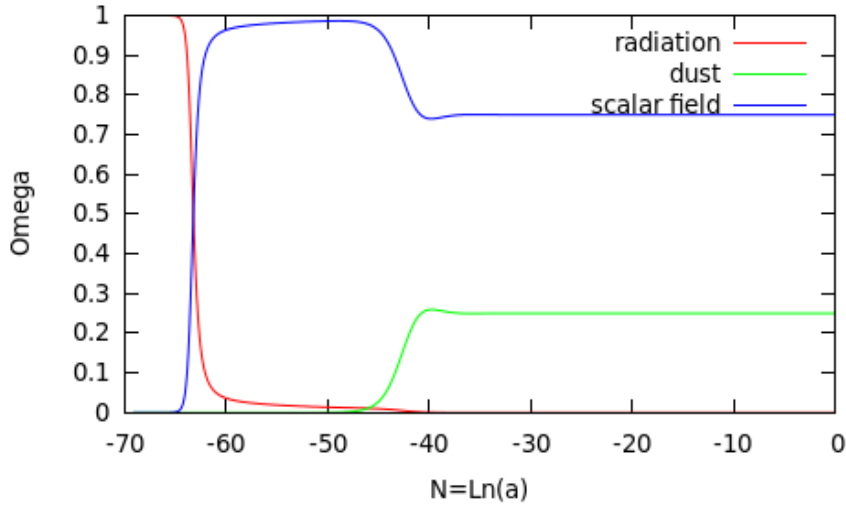


Figure 2.10: Evolution of dynamical variables, when $\lambda = 2$. Today's value of Hubble parameter for this case is: $H_0 \simeq 4.1 \times 10^{-10} S^{-1}$, that is better than previous case, but again very large compared to the observational value; and matter appears in this universe.

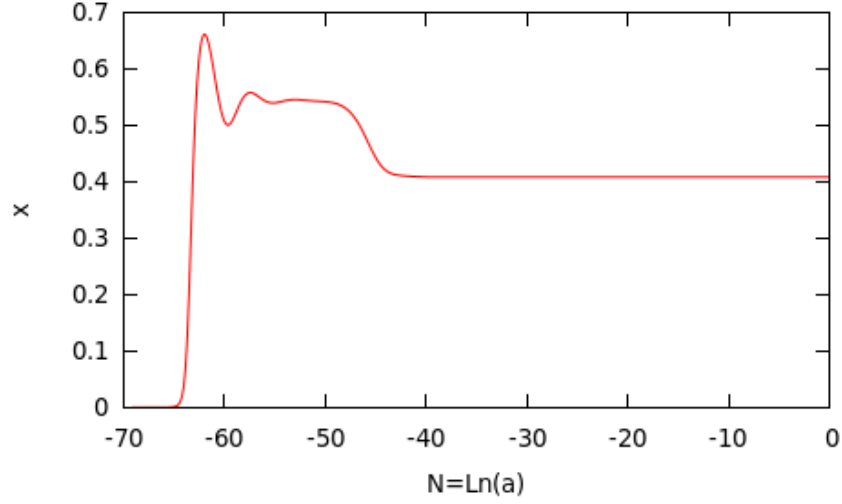


Figure 2.11: Evolution of kinetic energy of scalar field, x , when $\lambda = 3$.

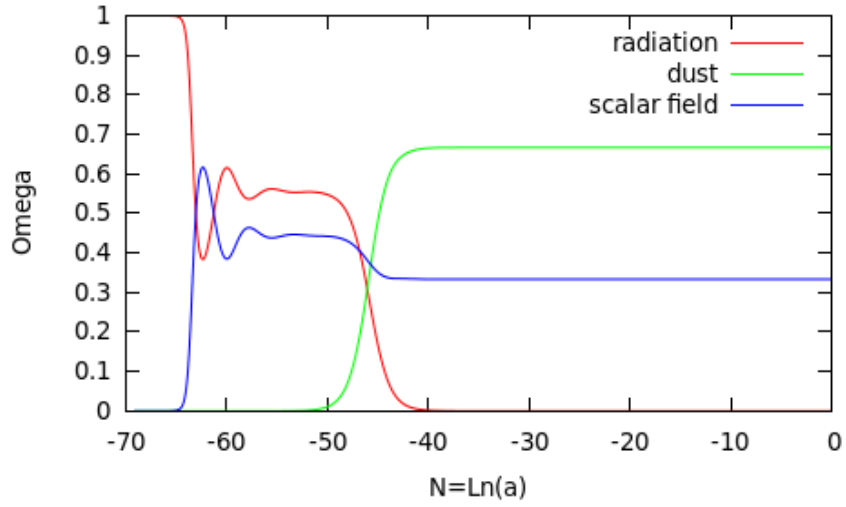


Figure 2.12: Evolution of dynamical variables, when $\lambda = 3$. Today's value of Hubble parameter for this case is: $H_0 \simeq 2.5 \times 10^{-10} S^{-1}$, that is better than previous cases, but again very large compared to the observational value.

Chapter 3

Analysis of Cosmological Two-Field

3.1 Two-Field System

As discussed in the first chapter according to the general relativity, gravity is a manifestation of the geometry of space-time and the gravitational degrees of freedom are described by the metric that is covariantly conserved under coordinate transformation; this is what is called the standard theory of gravity; for this case, the equations of motion was the Einstein field equation. But there are alternative theories of relativity with different approaches to gravity. Some of them are modifications of the gravity action and/or matter action[26]; they try to explain the puzzling experimental issues such as dark matter, dark energy and so on (the identities assumed to sustain general relativity and the Big Bang model of universe); and some of these alternative theories try to find a unified field theory[27], or answer to the theoretical issues such as the nature of time and space[28].

As one of these alternative theories, in non-metric gravity (or non-metric extension of the Einstein's general theory of relativity), the metric is no longer covariantly conserved under coordinate transformations, and space-time connection is independent of the metric; so the metric is not the only degree of freedom and the universe in addition to the background fluids, contains two fields: homogeneous scalar field (or scalar matter field), Ψ , and the graviscalar field, φ . The scalar matter field coupled not only to the metric but also to the graviscalar field; and the most general second order theory for Ψ with quadratic kinetic terms can be written as[18] :

$$2L = \frac{1}{\kappa^2}R - (\partial\varphi)^2 - (\partial\Psi)^2 + 2\gamma\kappa\Psi\Psi_{,\alpha}\varphi^{,\alpha} - V(\Psi) \quad (3.1)$$

Where R is the Ricci scalar; γ is an unknown coefficient that is constant and

$\kappa^2 = 8\pi G = \frac{1}{M_P^2}$, and M_P is the Planck mass. First Friedmann's equation for this system can be written as:

$$H^2 = \frac{\kappa^2}{3}\rho = \frac{\kappa^2}{3} \left[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\dot{\Psi}^2 - \gamma\kappa\Psi\dot{\Psi}\dot{\varphi} + V(\Psi) \right]$$

We can separate two fields By defining new variable v , as:

$$v = \varphi - \frac{1}{2}\gamma\kappa\Psi^2 \quad (3.2)$$

The effective lagrangian density of two-field system defined by (Ψ, v) , can be written as:

$$2L = \frac{1}{\kappa^2}R - (\partial v)^2 - (1 - \gamma^2\kappa^2\Psi^2)(\partial\Psi)^2 - V(\Psi) \quad (3.3)$$

From this equation, we can write energy density and pressure of the two-field system, as [18]:

$$\rho_v = \frac{1}{2}\dot{v}^2 \quad \text{and} \quad \rho_\Psi = \frac{1}{2}(1 - \gamma^2\kappa^2\Psi^2)\dot{\Psi}^2 + V(\Psi) \quad (3.4)$$

$$P_v = \frac{1}{2}\dot{v}^2 \quad \text{and} \quad p_\Psi = \frac{1}{2}(1 - \gamma^2\kappa^2\Psi^2)\dot{\Psi}^2 - V(\Psi) \quad (3.5)$$

And the first Friedmann equation for this system will be :

$$H^2 = \frac{\kappa^2}{3}\rho = \frac{\kappa^2}{3} \left[\frac{1}{2}\dot{v}^2 + \frac{1}{2}(1 - \gamma^2\kappa^2\Psi^2)\dot{\Psi}^2 + V(\Psi) \right] \quad (3.6)$$

Now the continuity equation will give the equations of motion of fields contributed in two-field system.

Continuity equations of Ψ -field and v -field can be written as:

$$\dot{\rho}_\Psi + 3H(\rho_\Psi + p_\Psi) = 0$$

$$\dot{\rho}_v + 3H(\rho_v + p_v) = 0$$

By substituting ρ_Ψ , p_Ψ , ρ_v and p_v , from equations (3.4) and (3.5), we can find the equations of motion of Ψ -field and v -field, as:

$$\ddot{\Psi} + 3H\dot{\Psi} - \frac{\gamma^2\kappa^2}{1 - \gamma^2\kappa^2\Psi^2}\Psi\dot{\Psi}^2 + \frac{V_{,\Psi}}{1 - \gamma^2\kappa^2\Psi^2} = 0 \quad (3.7)$$

$$\ddot{v} + 3H\dot{v} = 0 \quad (3.8)$$

Here we can find equation of motion of φ -field (similar to Ψ -field), by substituting $v = \varphi - \frac{1}{2}\gamma\kappa\Psi^2$ in equatin above:

$$\ddot{\varphi} - \gamma\kappa(\dot{\Psi}^2 + \Psi\ddot{\Psi}) + 3H(\dot{\varphi} - \gamma\kappa\Psi\dot{\Psi}) = 0$$

$\ddot{\Psi}$ can be substituted from equation (3.7), which gives:

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\gamma\kappa}{1 - \gamma^2\kappa^2\Psi^2}\dot{\Psi}^2 + \frac{\gamma\kappa\Psi V_{,\Psi}}{1 - \gamma^2\kappa^2\Psi^2} = 0 \quad (3.9)$$

Now in order to simplify our equations, we can define new variables for two different regions of values of Ψ as:

$$du_- = \sqrt{1 - \gamma^2\kappa^2\Psi^2} d\Psi, \quad \text{for the region } \Psi < \frac{1}{\gamma\kappa} \quad (3.10)$$

$$du_+ = \sqrt{\gamma^2\kappa^2\Psi^2 - 1} d\Psi, \quad \text{for the region } \Psi > \frac{1}{\gamma\kappa} \quad (3.11)$$

So, the equation of motion (Klein-Gordon equation) of u-field, can be written as:

$$\ddot{u}_{\mp} + 3H\dot{u}_{\mp} \pm \frac{V_{,\Psi(u_{\mp})}}{\sqrt{|1 - \gamma^2\kappa^2\Psi^2(u_{\mp})|}} = 0 \quad (3.12)$$

And the first Friedmann equation for the system of u-field and v-field, will be:

$$H^2 = \frac{\kappa^2}{3} \left[\frac{1}{2}\dot{v}^2 \pm \frac{1}{2}\dot{u}_{\mp}^2 + V(u_{\mp}) \right] \quad (3.13)$$

From the above equation, the energy density and pressure of u-field and v-field, can be written as:

$$\rho_v = \frac{1}{2}\dot{v}^2 \quad \text{and} \quad \rho_u = \pm \frac{1}{2}\dot{u}_{\mp}^2 + V(u_{\mp}) \quad (3.14)$$

$$p_v = \frac{1}{2}\dot{v}^2 \quad \text{and} \quad p_u = \pm \frac{1}{2}\dot{u}_{\mp}^2 - V(u_{\mp}) \quad (3.15)$$

Here we can conclude that equations of motion (or Klein-Gordon equations) of two-field system, can be defined in three different ways, using different variables:

- System of φ and Ψ fields (equations 3.7 and 3.9)
- System of v and Ψ fields (equations 3.7 and 3.8)
- System of v and u_{\mp} fields (equations 3.8 and 3.12)

3.2 Dimensionless Variables

Now we can define dimensionless variables of two-field system, defined by u_{\mp} and v fields, as:

$$x_{\mp} = \sqrt{\frac{\kappa^2 \dot{u}_{\mp}^2}{6H^2}} \quad , \quad y = \sqrt{\frac{\kappa^2 V(u_{\mp})}{3H^2}} \quad , \quad z = \sqrt{\frac{\kappa^2 \dot{v}^2}{6H^2}} \quad (3.16)$$

constrained by first Friedmann equation:

$$z^2 + x_{\mp}^2 + y^2 = 1 \quad , \quad \text{if } \Psi < \frac{1}{\gamma\kappa} \quad (3.17)$$

$$z^2 - x_{\mp}^2 + y^2 = 1 \quad , \quad \text{if } \Psi > \frac{1}{\gamma\kappa} \quad (3.18)$$

In order to study the evolution of the two-field system, we derive the derivatives of these dimensionless variables with respect to logarithm of scale factor, $N = \ln(a)$:

$$x'_{\mp} = \frac{1}{H} \dot{x}_{\mp} = \frac{\kappa}{\sqrt{6}H} \left(\frac{1}{H} \ddot{u}_{\mp} - \frac{\dot{H}}{H^2} \dot{u}_{\mp} \right) \quad (3.19)$$

$$y' = \frac{1}{H} \dot{y} = \frac{\kappa}{\sqrt{3}H} \left(\frac{1}{2H} \frac{V_{,\Psi}(u_{\mp}) \dot{\Psi}(u_{\mp})}{\sqrt{V(u_{\mp})}} - \frac{\dot{H}}{H^2} \sqrt{V} \right) \quad (3.20)$$

$$z' = \frac{1}{H} \dot{z} = \frac{\kappa}{\sqrt{6}H} \left(\frac{1}{H} \ddot{v} - \frac{\dot{H}}{H^2} \dot{v} \right) \quad (3.21)$$

$\frac{\dot{H}}{H^2}$ can be found from equation (3.13) as:

$$\frac{\dot{H}}{H^2} = \frac{\kappa^2}{6H^3} \left[\dot{v} \ddot{v} \pm \dot{u}_{\mp} \ddot{u}_{\mp} + V_{,\Psi} \dot{\Psi} \right]$$

$\dot{\Psi}$ can be substituted from relations (3.10) and (3.11); \ddot{v} from equation (3.8) and \ddot{u}_{\mp} from (3.12). So we have:

$$\frac{\dot{H}}{H^2} = \frac{\kappa^2}{6} \left[-3 \frac{\dot{v}^2}{H^2} \mp 3 \frac{\dot{u}_{\mp}^2}{H^2} \right]$$

And in terms of dimensionless variables, it will be:

$$\frac{\dot{H}}{H^2} = -3(1 - y^2) \quad (3.22)$$

From now on, we use power-law potential for this system, defined as:

$$V \propto \Psi^{\alpha} \quad (3.23)$$

And by substituting $\frac{\dot{H}}{H^2}$ from equation (3.22), and \ddot{v} and \ddot{u}_\mp , from Klein-Gordon equations (equations 3.8 and 3.12), in equations (3.19) to (3.21), we can find the dynamics of dimensionless variables as:

$$x'_\mp = -3x_\mp y^2 \mp \sqrt{\frac{3}{2}} \frac{\alpha y^2}{\kappa \Psi \sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_\mp)|}} \quad (3.24)$$

$$y' = 3y(1 - y^2) + \sqrt{\frac{3}{2}} \frac{\alpha y x_\mp}{\kappa \Psi \sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_\mp)|}} \quad (3.25)$$

$$z' = -3zy^2 \quad (3.26)$$

This system is non-autonomous, because of the existence of Ψ , in these equations.

3.2.1 Equations of State

The equation of state of this two-field system can be written as:

$$\gamma_{tot} = \frac{\rho_{tot} + p_{tot}}{\rho_{tot}} = 2(1 - y^2) \quad (3.27)$$

And the effective equation of state for the u_\mp -field and v -field, at any point are given by:

$$\gamma_{u_\mp} = \frac{\rho_{u_\mp} + p_{u_\mp}}{\rho_{u_\mp}} = \frac{\pm 2x_\mp^2}{\pm x_\mp^2 + y^2} \quad (3.28)$$

$$\gamma_v = \frac{\rho_v + p_v}{\rho_v} = \frac{2\rho_v}{\rho_v} = 2 \quad (3.29)$$

3.2.2 Regions of Validity of Dimensionless Variables

According to the constraint equations (3.17) and (3.18), density parameters of u_\mp and v fields are given by:

$$\Omega_{u_\mp} = \pm x_\mp^2 + y^2 \quad \text{and} \quad \Omega_v = z^2 \quad (3.30)$$

From the equation of motion of v -field, equation (3.8), one can find its evolution in terms of scale factor, as:

$$\ddot{v} + 3H\dot{v} = 0 \implies \dot{v}(a) \propto a^{-3} \quad (3.31)$$

So its corresponding energy density, $\rho_v = \frac{1}{2}\dot{v}^2$, is always positive and we have:

$$z^2 = \frac{\kappa^2 \rho_v}{3H^2} > 0$$

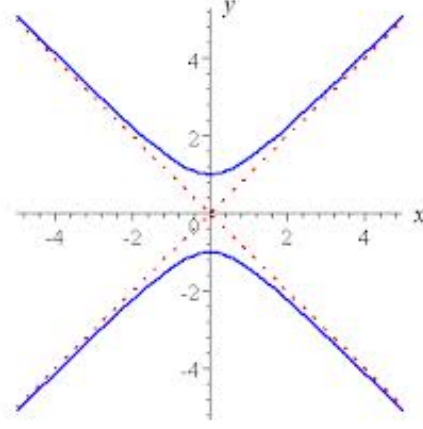


Figure 3.1: x_+ and y vary inside the region between upper and lower branches of a hyperbola defined by $y^2 - x_+^2 = 1$.

And for the positive potential $V(\Psi)$, corresponding dimensionless variable $y^2 = \frac{\kappa^2 V(\Psi)}{3H^2} > 0$; and from the relation $\dot{u}_\mp^2 = |1 - (\gamma\kappa\Psi)^2|\dot{\Psi}^2$, we can see that \dot{u}_\mp^2 is positive, so $x_\mp^2 = \frac{\kappa^2 \dot{u}_\mp^2}{6H^2} > 0$.

Now the region of validity of dimensionless variables describing scalar field Ψ , can be found for two different regions of Ψ as:

1- $\Psi < \frac{1}{\gamma\kappa}$: for this case, we have:

$$0 \leq z^2 \leq 1 \quad \text{and} \quad 0 \leq y^2 + x_-^2 \leq 1 \quad (3.32)$$

So x_- and y are constrained within a disc of unit radius:

$$-1 \leq x_- \leq 1$$

$$-1 \leq y \leq 1$$

This case is similar to scalar field discussed in chapter 2.

2- $\Psi > \frac{1}{\gamma\kappa}$: for this case also, the constraint equation demands that:

$$y^2 - x_+^2 + z^2 = 1 \implies y^2 - x_+^2 \leq 1 \quad (3.33)$$

So x_+ and y evolve within a hyperbola, as shown in figure (3.1); and all the variables can be larger than $|\pm 1|$. Density parameter for this case, Ω_{u+} can be negative, too.

3.3 Non-autonomous Phase Diagrams

The system of dimensionless variables defined for two-field of $u_{\mp}(\Psi)$ and $v(\Psi, \varphi)$, is non-autonomous:

$$\begin{aligned}x' &= x'(x, y, \Psi) \\ y' &= y'(x, y, \Psi) \\ z' &= z'(z, y)\end{aligned}$$

Although (x', y') cannot be written in terms of x and y , only, But their phase diagrams can be found, numerically. The phase planes of (x', y') (or u -field), will be meaningless and the trajectories will cross each other and project into an indecipherable mess[29].

Figures (3.2) and (3.5) are phase planes of u_{\mp} for $\Psi < \frac{1}{\gamma\kappa}$ and $\Psi > \frac{1}{\gamma\kappa}$. We cannot talk about symmetry of phase planes from equations (3.24) and (3.25), because of the existence of Ψ in these equations; but the planes themselves (Although meaningless), show symmetries about x and y axis for both two different region of Ψ -field.

One cannot get more information about the universe, from these diagrams and in order to find the evolution of the universe and its components, the system of dimensionless variables must be solved.

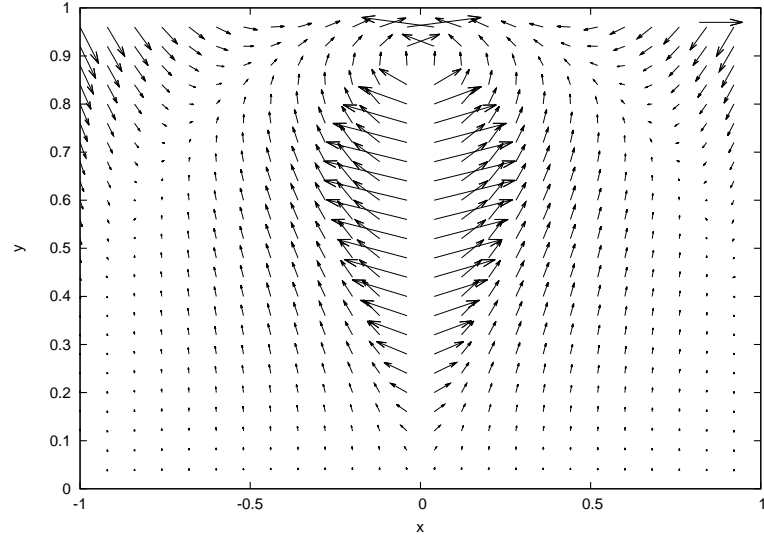


Figure 3.2: The phase plane of u_- -field when $\Psi < \frac{1}{\gamma\kappa}$, in upper half-disc ($y > 0$).

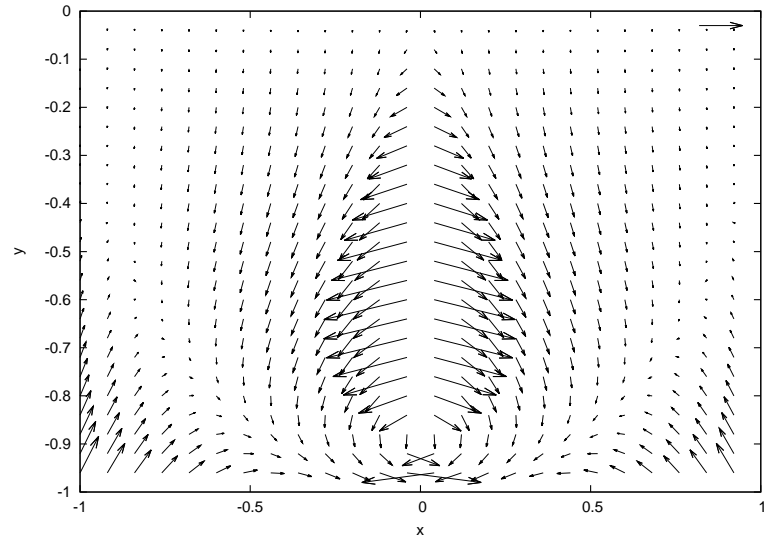


Figure 3.3: The phase plane of u_- -field when $\Psi < \frac{1}{\gamma\kappa}$, in lower half-disc ($y < 0$).

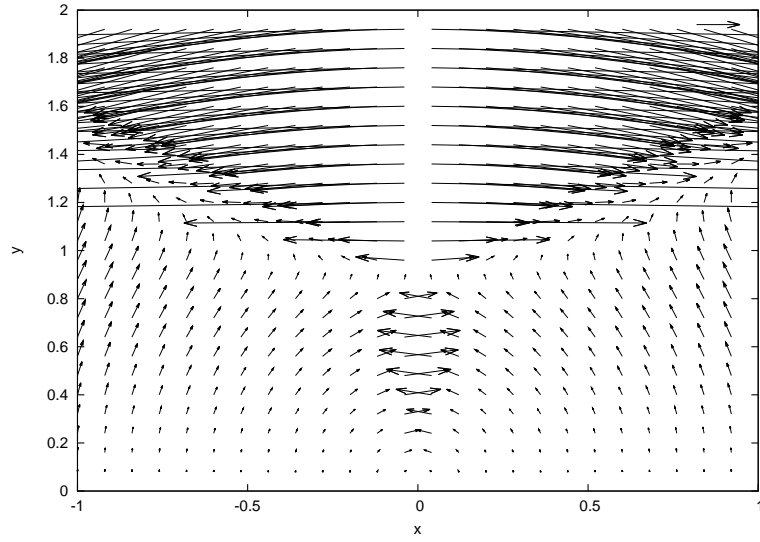


Figure 3.4: The phase plane of u_+ -field when $\Psi > \frac{1}{\gamma\kappa}$, for $y > 0$.

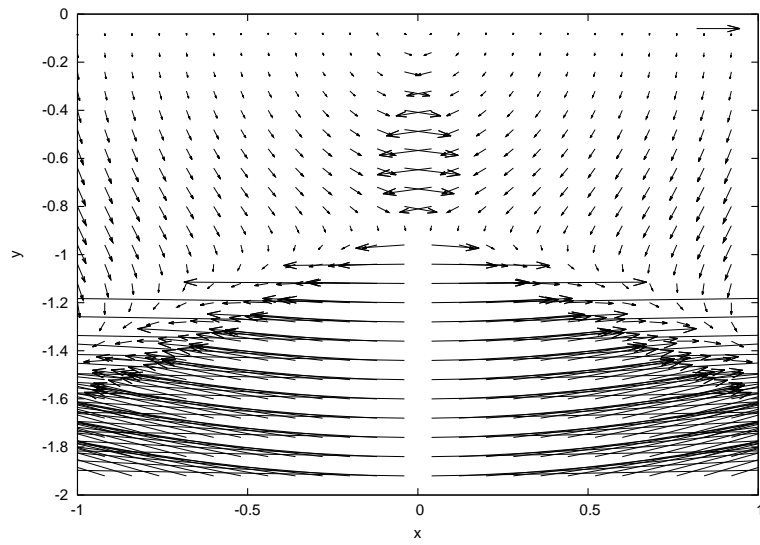


Figure 3.5: The phase plane of u_+ -field when $\Psi > \frac{1}{\gamma\kappa}$, for $y < 0$.

3.4 Evolution of Universe filled with Two-Field and Background Fluids

3.4.1 Dimensionless Variables and their Numerical Solutions

For a spatially-flat FRW model of universe containing radiation, matter (baryonic and dark matter), v -field and u_{\mp} -field, first Friedmann's equation (or constraining equation) can be written as:

$$H^2 = \frac{\kappa^2}{3} \left[\frac{1}{2} \dot{v}^2 \pm \frac{1}{2} \dot{u}_{\mp}^2 + V(u_{\mp}) + \rho_r + \rho_m \right] \quad (3.34)$$

According to their definitions in chapters 2 and 3, dimensionless variables (or density parameters) for this case are as follows:

$$x_{\mp}^2 = \frac{\kappa^2 \dot{u}_{\mp}^2}{6H^2} \quad , \quad y^2 = \frac{\kappa^2 V(u_{\mp})}{3H^2} \quad , \quad z^2 = \frac{\kappa^2 \dot{v}^2}{6H^2} \quad , \quad r^2 = \frac{\kappa^2 \rho_r}{3H^2} \quad , \quad m^2 = \frac{\kappa^2 \rho_m}{3H^2}$$

And $\frac{\dot{H}}{H^2}$ will be written as:

$$\frac{\dot{H}}{H^2} = -3 + 3y^2 + r^2 + \frac{3}{2}m^2$$

And the system of non-autonomous equations, defining the evolution of dimensionless variables in terms of logarithm of scale factor, will be:

$$x'_{\mp} = -x_{\mp}(3y^2 + r^2 + \frac{3}{2}m^2) \mp \sqrt{\frac{3}{2}} \frac{\alpha y^2}{\kappa \Psi \sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_{\mp})|}} \quad (3.35)$$

$$y' = y(3 - 3y^2 - r^2 - \frac{3}{2}m^2) + \sqrt{\frac{3}{2}} \frac{\alpha y x_{\mp}}{\kappa \Psi \sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_{\mp})|}} \quad (3.36)$$

$$z' = -z(3y^2 + r^2 + \frac{3}{2}m^2) \quad (3.37)$$

$$r' = r(1 - 3y^2 - r^2 - \frac{3}{2}m^2) \quad (3.38)$$

$$m' = m(\frac{3}{2} - 3y^2 - r^2 - \frac{3}{2}m^2) \quad (3.39)$$

These equations are derived for adiabatically expanding universe, assuming that the potential of Ψ -field, $V(\Psi)$, is power-law. The monomial definition of $V(\Psi)$ can be written as [18]:

$$V(\Psi) = \lambda \kappa^{\alpha-4} \Psi^{\alpha} \quad (3.40)$$

where λ is a constant.

The non-autonomous system of equations of motion can be solved numerically; but first it must be converted to an autonomous system. In order to

do that, I have to define another equation as:

$$\Psi' = \frac{1}{H} \dot{\Psi} = \frac{\sqrt{6}x_{\mp}}{\kappa\sqrt{|1 - \gamma^2\kappa^2\Psi^2(u_{\mp})|}} \quad (3.41)$$

This equation plus five equations of non-autonomous system, make an autonomous system of six equations.

As a initial conditions we can assume that v-field dominates early universe, instead of radiation which dominated early universe (containing scalar field, radiation and matter) discussed in chapter two; this is because of the density of v-field that is proportional to a^{-6} , while the radiation density is proportional to a^{-4} ; and y is very small for early universe, assuming that $V(\Psi)$ is very small at very early universe and Hubble parameter is very large at that times; the density of u-field, $\frac{1}{2}\dot{u}^2$, at early universe, is also small compared to the density of v-field, but it may be not negligible.

For two different values of λ , γ and α , the evolution of density parameters are shown in figures (3.6) and (3.7); in these figures u-field represents the density parameter of kinetic energy of u-field.

In figures (3.6) and (3.7), dimensionless variable y dominates today's universe which causes the domination of negative pressure, responsible for accelerated expansion of the universe today; but today's value of Hubble parameter is about $69.32 \pm 0.80 \text{ Km S}^{-1} \text{ Mpc}^{-1} \simeq 2.3 \times 10^{-18} \text{ S}^{-1}$ (according to the measurements [30]), that is very smaller than the values found for these cases, $\sim 3 \times 10^{-10} \text{ S}^{-1}$ and $\sim 4.4 \times 10^{-10} \text{ S}^{-1}$, respectively.

According to the figure (3.6), Radiation and matter never dominate universe; and in figure (3.7), matter does not appear and radiation appear in the future, that is not realistic. So these figures, started from v-field dominated early universe, are not compatible with our universe.

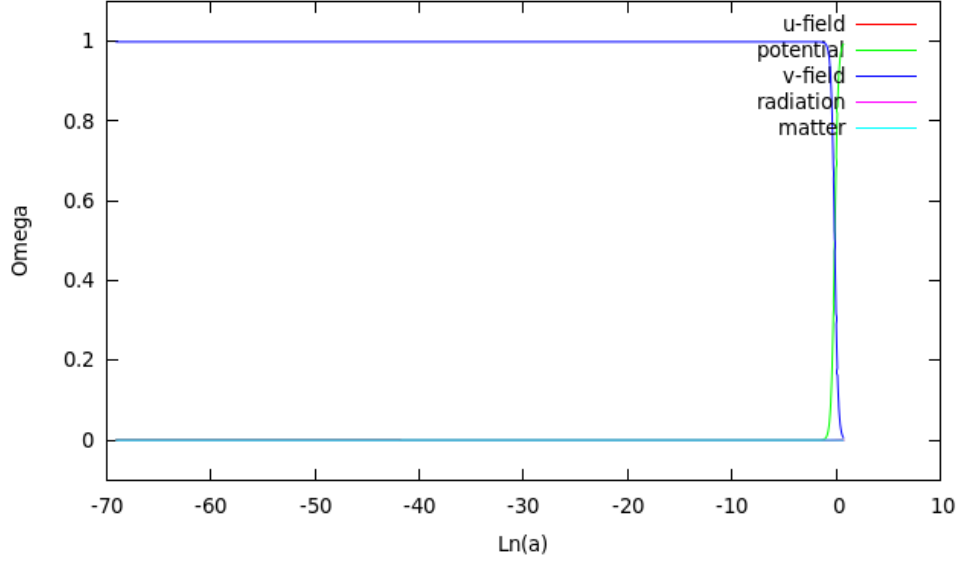


Figure 3.6: Evolution of dynamical variables when $\lambda = 2 \times 10^{-30}$, $\alpha = 2$, $\gamma = 0.08$; starting from initial values, $x_{init.}^2 = 10^{-3}$, $z_{init.}^2 = 0.999$, $y_{init.} = 10^{-90}$ and the coefficient of density of v-field is: 2×10^{-10} . From $\text{Ln}(a) = -70$ to about -41, this system is in minus region ($\Psi < \frac{1}{\gamma\kappa}$), but after that it switches to plus region ($\Psi > \frac{1}{\gamma\kappa}$); it happens at $\Psi \simeq 305639$. According to this system today's Hubble parameter is about $3 \times 10^{-10} S^{-1}$, that is very large number compared to the observational value, $H_0 \simeq 2.3 \times 10^{-18} S^{-1}$. Radiation and matter never dominate this universe.

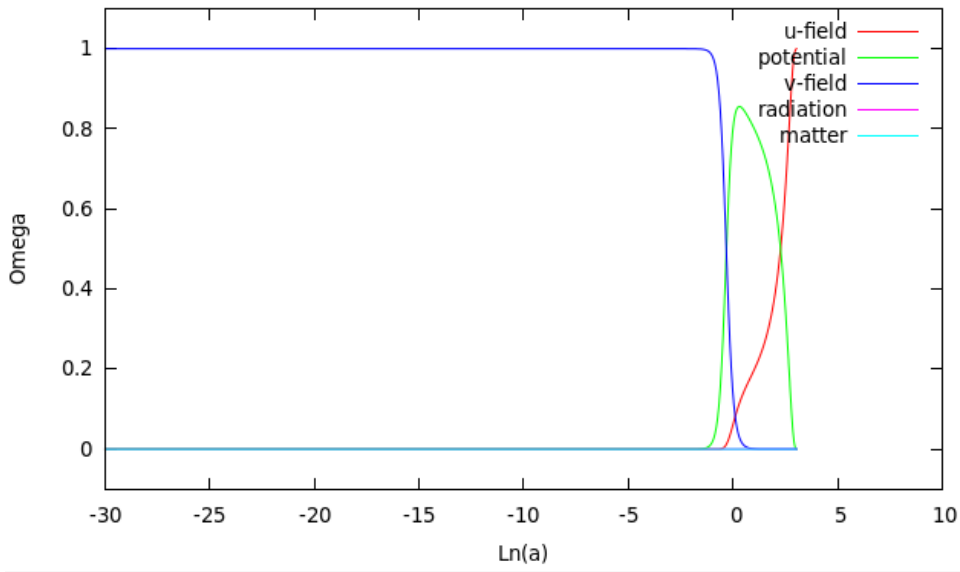


Figure 3.7: Evolution of dynamical variables when $\lambda = 2 \times 10^{-30}$, $\alpha = 4$, $\gamma = 0.1$; starting from initial values, $x_{init.}^2 = 10^{-4}$, $z_{init.}^2 = 0.9999$, $y_{init.} = 10^{-90}$ and the coefficient of density of v-field is: 2×10^{-10} . For this case Ψ stays in minus region all the time. Today's Hubble parameter for this universe is: $H_0 \simeq 4.4 \times 10^{-10} S^{-1}$, that is a very large number compared to the observational value.

In order to have a more realistic universe with matter and radiation dominated eras, one should assume that the very early universe was dominated by kinetic energy density of u-field, $\frac{1}{2}\dot{u}^2$, and the energy density of v-field, $\frac{1}{2}\dot{v}^2$, was small. As figures (3.8) and (3.11) show, the kinetic energy of u-field dominates early universe which in the later times, it gives its place to radiation, matter and the potential energy of u-field, respectively. These plots are found, after separating energy density parameters of baryonic matter and dark matter from each other, and adding one more equation to the system of autonomous equations; so we have:

$$m^2 = m_{dark}^2 + m_{baryonic}^2$$

and instead of equation (3.39), we have these two equations:

$$\begin{aligned} m'_{dark} &= m_{dark} \left(\frac{3}{2} - 3y^2 - r^2 - \frac{3}{2}m^2 \right) \\ m'_{baryonic} &= m_{baryonic} \left(\frac{3}{2} - 3y^2 - r^2 - \frac{3}{2}m^2 \right) \end{aligned}$$

According to the plots, transition from minus region to the plus region of Ψ -field happens when radiation starts to dominate our universe (that is $Ln(a) \simeq -36.8$). These pictures are more compatible with the history of our universe and observations (measured values of today's Hubble parameter, H_0 , today's energy density parameters of dark matter, Ω_m , baryonic matter, Ω_b , and dark energy, Ω_Λ ; and the age and size of the universe which will be discussed later in this chapter). According to these pictures v-field has nothing to do with our universe; its corresponding dimensionless variable, z , starts from a small value and becomes more smaller through the time.

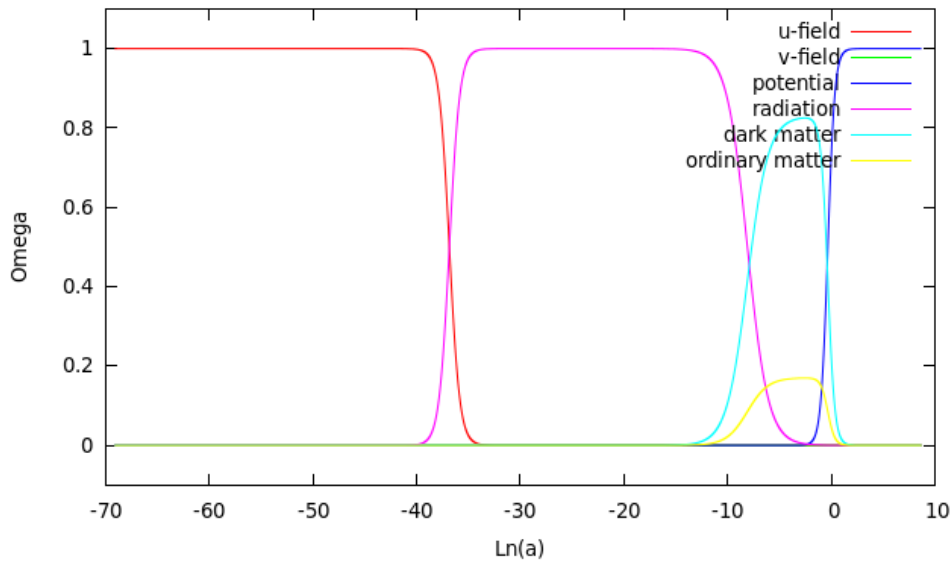


Figure 3.8: Evolution of dynamical variables when $\lambda = 1.8 \times 10^{-48}$, $\alpha = 2$, $\gamma = 0.01$; starting from initial values, $x_{init.} = 1$, $y_{init.} = 10^{-80}$ and $r_{init.} = 10^{-14}$. Today's value of Hubble parameter for this universe is: $H_0 \simeq 2.26 \times 10^{-18} S^{-1}$ and the age of the universe is: $4.37 \times 10^{17} S$. Dark energy, dark Matter and baryonic matter density parameters of today's universe are: $\Omega_\Lambda \simeq 0.727$, $\Omega_m \simeq 0.226$ and $\Omega_b \simeq 0.046$.

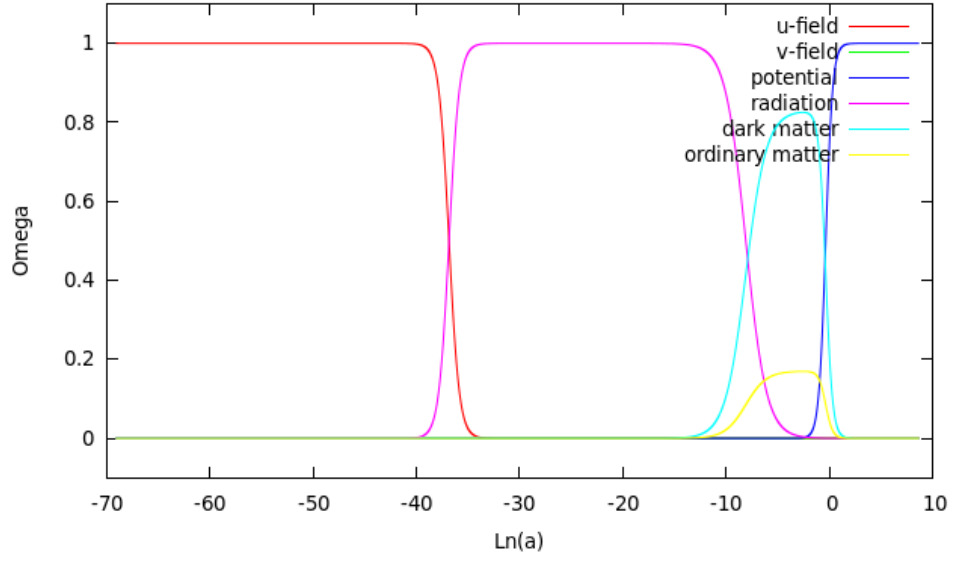


Figure 3.9: Evolution of dynamical variables when $\lambda = 1.8 \times 10^{-52}$, $\alpha = 4$, $\gamma = 0.01$; starting from initial values, $x_{init.} = 1$, $y_{init.} = 10^{-78}$ and $r_{init.} = 10^{-14}$. Today's value of Hubble parameter for this universe is: $H_0 \simeq 2.31 \times 10^{-18} S^{-1}$ and the age of the universe is: $4.33 \times 10^{17} S$. Dark energy, dark Matter and baryonic matter density parameters of today's universe are: $\Omega_\Lambda \simeq 0.735$, $\Omega_m \simeq 0.22$ and $\Omega_b \simeq 0.045$.

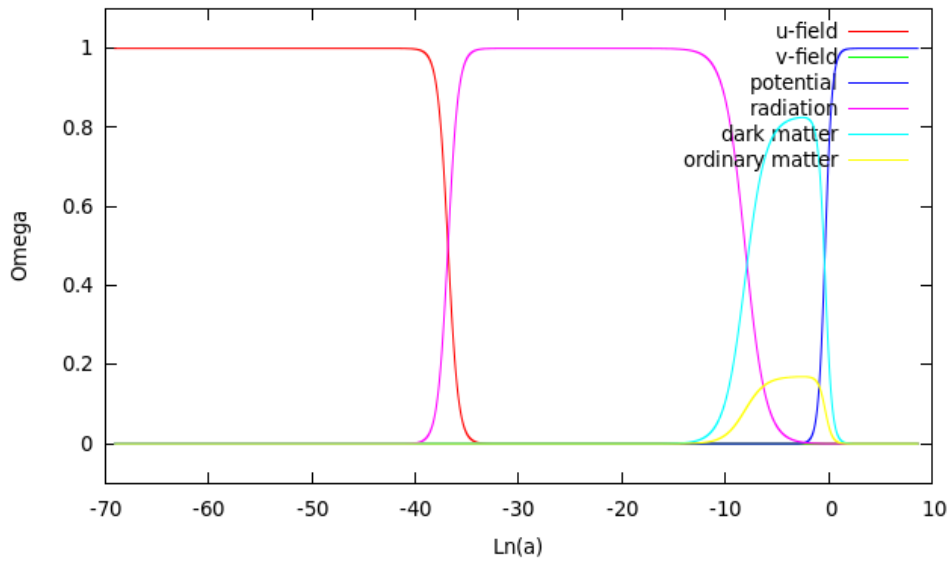


Figure 3.10: Evolution of dynamical variables when $\lambda = 5.2 \times 10^{-61}$, $\alpha = 8$, $\gamma = 0.01$; starting from initial values, $x_{init.} = 1$, $y_{init.} = 10^{-78}$ and $r_{init.} = 10^{-14}$. Today's value of Hubble parameter for this universe is: $H_0 \simeq 2.18 \times 10^{-18} S^{-1}$ and the age of the universe is: $4.35 \times 10^{17} S$. Dark energy, dark Matter and baryonic matter density parameters of today's universe are: $\Omega_\Lambda \simeq 0.683$, $\Omega_m \simeq 0.268$ and $\Omega_b \simeq 0.049$.

3.4.2 Variation of Potential $V(u)$

The potential V is defined as a function of Ψ , and the evolution of $V(Ln(a))$ and $\Psi(Ln(a))$ are found numerically; but two-field components of our universe are u_{\mp} and v fields; so in order to find the variation of potential V in terms of u_{\mp} , one needs to find the values of $u_{\mp}(\Psi)$ for any given Ψ . u at minus and plus regions can be found from equations (3.10) and (3.11), as functions of Ψ :

$$u_- = \Psi \frac{\sqrt{1 - \gamma^2 \kappa^2 \Psi^2}}{2} + \frac{\arcsin(\gamma \kappa \Psi)}{2\gamma \kappa}, \quad (\Psi < \frac{1}{\gamma \kappa}) \quad (3.42)$$

$$u_+ = \frac{\pi}{4\gamma \kappa} + \Psi \frac{\sqrt{\gamma^2 \kappa^2 \Psi^2 - 1}}{2} - \frac{Ln(\gamma \kappa \Psi + \sqrt{\gamma^2 \kappa^2 \Psi^2 - 1})}{2\gamma \kappa}, \quad (\Psi > \frac{1}{\gamma \kappa}) \quad (3.43)$$

But it also can be found numerically by adding equation:

$$u'_{\mp} = \frac{du_{\mp}}{dN} = \frac{\dot{u}_{\mp}}{H} = \frac{\sqrt{6}}{\kappa} x_{\mp} \quad (3.44)$$

to the system of autonomous equations. Using data of u found from equation above, and the corresponding data of potential density $V(\Psi)$ (found from the numerical solution of Ψ), the plots of $V(u)$, are found.

Figures (3.11)through (3.13) show $V(u)$ for the cases dominated by kinetic energy of u -field at early times (answers compatible with real universe). According to these pictures, potential increases as u increases; and there is a shift from minus region to the plus region at $u \simeq 1917970$ (or $\gamma \kappa u \simeq 0.785 \simeq \frac{\pi}{4}$). Today's values of potential density found for these cases (which they are mass-energy density of dark energy), are compatible with its value calculated from observational values of Hubble parameter and dark energy density parameter:

$$V(t_0) = \rho_{\Lambda 0} = \frac{3H_0^2 \Omega_{\Lambda 0}}{\kappa^2} \simeq 5.81 \times 10^{-27} \text{ kg/m}^3$$

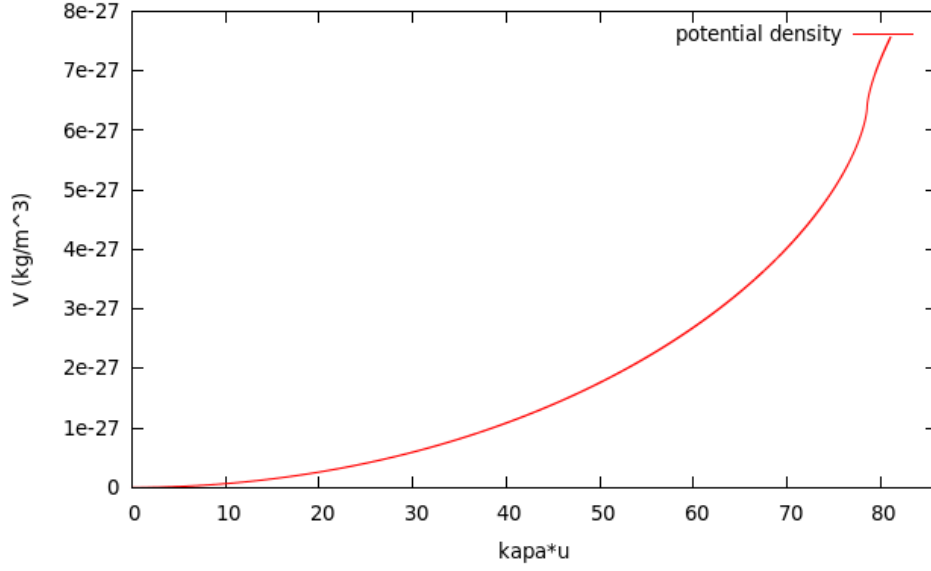


Figure 3.11: Variation of potential $V(u)$ for $\alpha = 2$, $\gamma = 0.01$ and $x_{init.} = 1$. Today's value of potential is about $6.63 \times 10^{-27} \text{ kg/m}^3$.

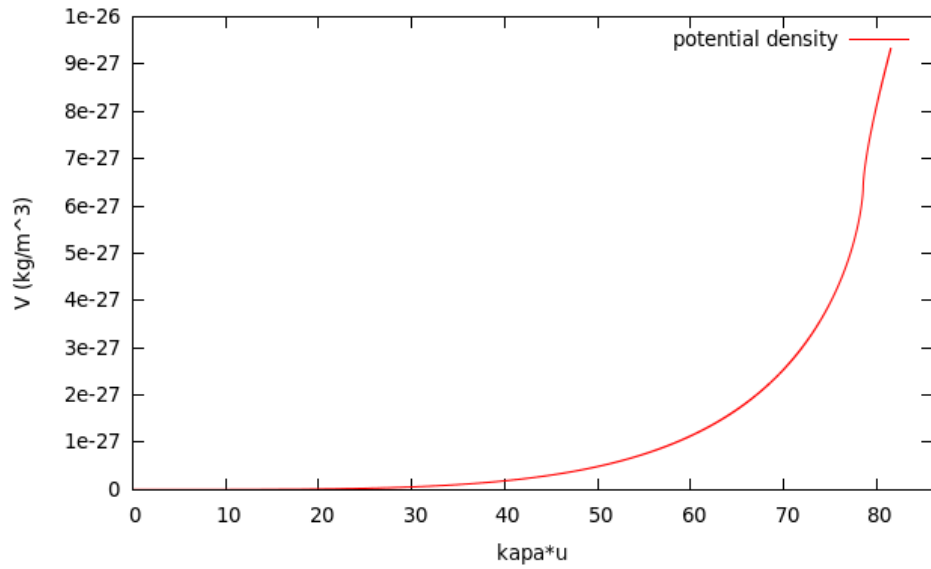


Figure 3.12: Variation of potential $V(u)$ for $\alpha = 4$, $\gamma = 0.01$ and $x_{init.} = 1$. Today's value of potential is about $6.9 \times 10^{-27} \text{ kg/m}^3$.

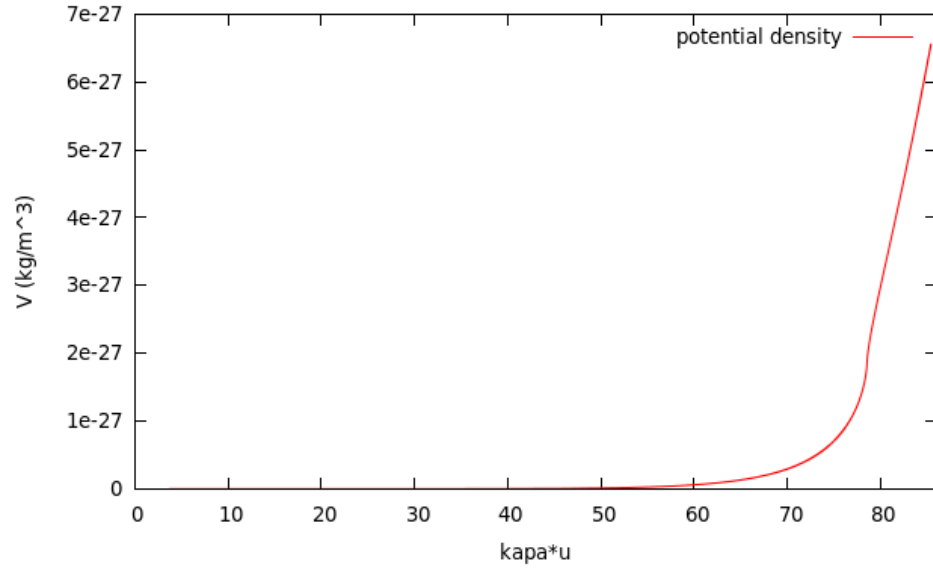


Figure 3.13: Variation of potential $V(u)$ for $\alpha = 8$, $\gamma = 0.01$ and $x_{init.} = 1$. Today's value of potential is about $5.8 \times 10^{-27} \text{ kg/m}^3$.

3.4.3 Hubble Parameter and Evolution of Scale Factor

The values of Hubble parameter over time, can be found numerically, by adding the equation below to the system of autonomous equations discussed before:

$$H' = \frac{dH}{dN} = \frac{\dot{H}}{H} = H(-3 + 3y^2 + r^2 + \frac{3}{2}m^2) \quad (3.45)$$

Figures (3.14) and (3.15), show the evolution of Hubble parameter and scale factor, a , for $\alpha = 8$. Hubble parameter decreases from a very large value to a very small value, as universe evolves; its today's value for this case is:

$$H_0 \simeq 2.18 \times 10^{-18} \text{ S}^{-1} \simeq 67.29 \text{ km/s/Mpc}$$

The evolution of scale factor over time, shows a rapid growth for early universe which turns to a linear growth later; the age of the universe can also be found from this picture:

$$t_0 = 4.3 \times 10^{17} \text{ s} \simeq 1.3 \times 10^{10} \text{ year}$$

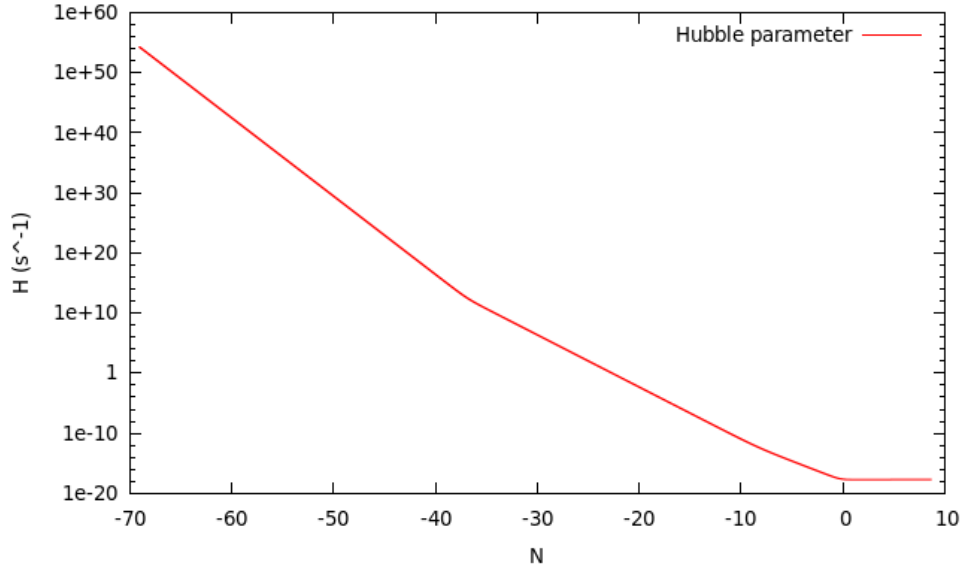


Figure 3.14: Evolution of Hubble parameter for $\alpha = 8$, $\gamma = 0.01$ and $x_{init.} = 1$.

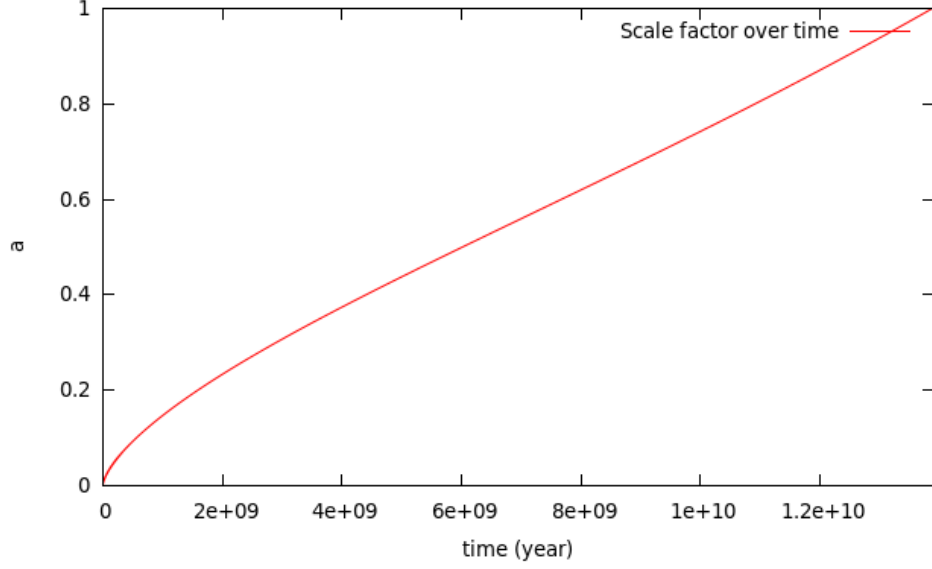


Figure 3.15: evolution of scale factor with respect to time for $\alpha = 8$, $\gamma = 0.01$ and $x_{init.} = 1$. The age of today's universe is about $4.3 \times 10^{17} \text{ s} \simeq 1.3 \times 10^{10} \text{ year}$.

Scale factor of early universe

According to the non-metric quintessential model of dark energy, our universe was dominated by kinetic energy of u-field, shown in figures (3.8) through (3.10); the equation of state of that era was, $w=1$, so the pressure of the universe was positive; may be that era was inside inflationary era and positive pressure of u-field was slowing down our inflationary universe. The first Friedmann's equation for this era can be written as:

$$H^2 \simeq \frac{\kappa^2}{6} \dot{u}^2 \implies \frac{da}{a} \simeq \frac{\kappa}{\sqrt{6}} du \implies a \propto e^{\frac{\kappa}{\sqrt{6}} u} \quad (3.46)$$

Where u is in minus region, and the size of our early universe is exponential function of u .

$w=1$ for this era means that:

$$\frac{1}{2} \dot{u}^2 \propto a^{-6} \implies du \propto \sqrt{2} a^{-3} dt \quad (3.47)$$

By substituting du from relation (3.46), one can find the evolution of scale factor of early universe in terms of time:

$$\frac{\sqrt{6}}{\kappa} \frac{da}{a} \propto \sqrt{2} a^{-3} dt \implies a \propto [a_i^3 + \sqrt{3} \kappa (t - t_i)]^{\frac{1}{3}} \quad (3.48)$$

Where t_i and a_i are the time and scale factor of our early universe at inflationary era.

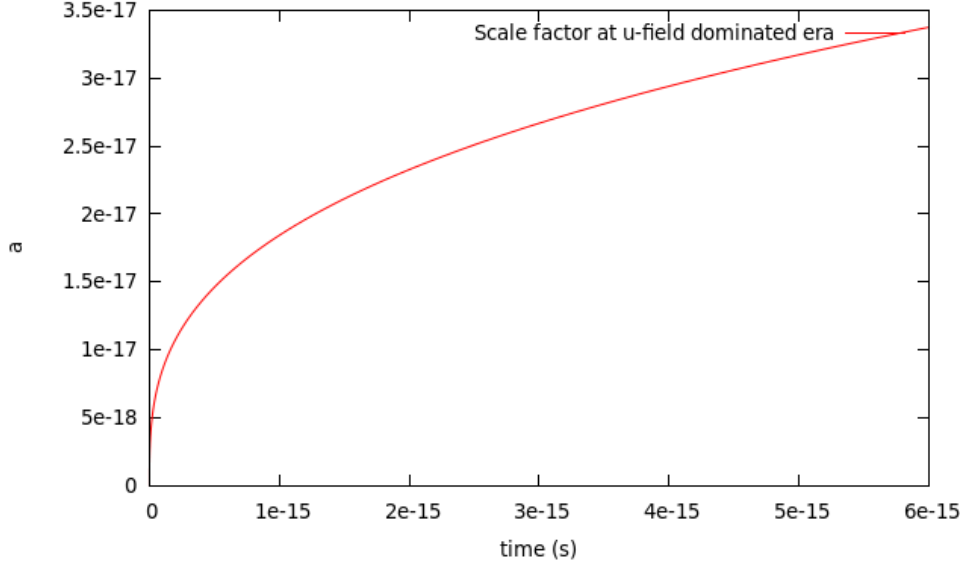


Figure 3.16: evolution of scale factor 'a' at early universe for $\alpha = 8$, $\gamma = 0.01$ and $x_{init.} = 1$.

Scale factor in the future

When the potential of two-field, $V(\Psi)$ dominates our universe ($w=-1$), that is going to happen in the future, according to the pictures (3.8) through (3.10), every things will be negligible compared to the negative pressure: $P = -V(\Psi)$. First Friedmann's equation for this era can be written as:

$$H^2 \simeq \frac{\kappa^2}{3} V(\Psi) = \frac{1}{3} \lambda \kappa^{\alpha-2} \Psi^\alpha \implies$$

$$a \propto e^{(\sqrt{\frac{\lambda}{3}} \kappa^{\frac{\alpha}{2}-1} \int \Psi^{\frac{\alpha}{2}} dt)} \quad (3.49)$$

3.4.4 The Size of the Universe

The size of the universe at any time, called conformal time $\eta(t)$, is the distance travelled by light since the Big Bang until that time; that is given by:

$$\eta(t) = \int_0^t \frac{cdt'}{a(t')} \quad \text{or} \quad \frac{d\eta}{dt} = \frac{c}{a} \quad (3.50)$$

$$\Rightarrow \quad \frac{d\eta}{dN} = \frac{1}{H} \frac{c}{a} \quad (3.51)$$

On the other hand, the proper distance of particle horizon of a universe starting from a Big Bang, as discussed in chapter 1, is given by:

$$d_p^{PH}(t) = a(t) \int_0^t \frac{cdt'}{a(t')} \\ \Rightarrow \quad \frac{d}{dt} \left[\frac{d_p^{PH}(t)}{a(t)} \right] = \frac{c}{a(t)} \quad \Rightarrow \quad \frac{d}{dN} [d_p^{PH}(t)] = d_p^{PH}(t) + \frac{c}{H} \quad (3.52)$$

$d_p^{PH}(t)$ is in fact the size of our today's visible universe, at any time in the past; but $\eta(t)$ (or comoving proper distance) is the size of particle horizon at any time.

Numerical solutions of these equations are shown in figure (3.17) for $\alpha = 8$; according to this picture the size of our today's universe, based on two-field model of dark energy, has always been smaller than the particle horizon in the past and its today's value is about:

$$d_p^{PH}(t_0) = \eta(t_0) \simeq 4.36 \times 10^{26} \text{ m} \simeq 14.14 \text{ Gigaparsecs}$$

That is the same as its measured value.

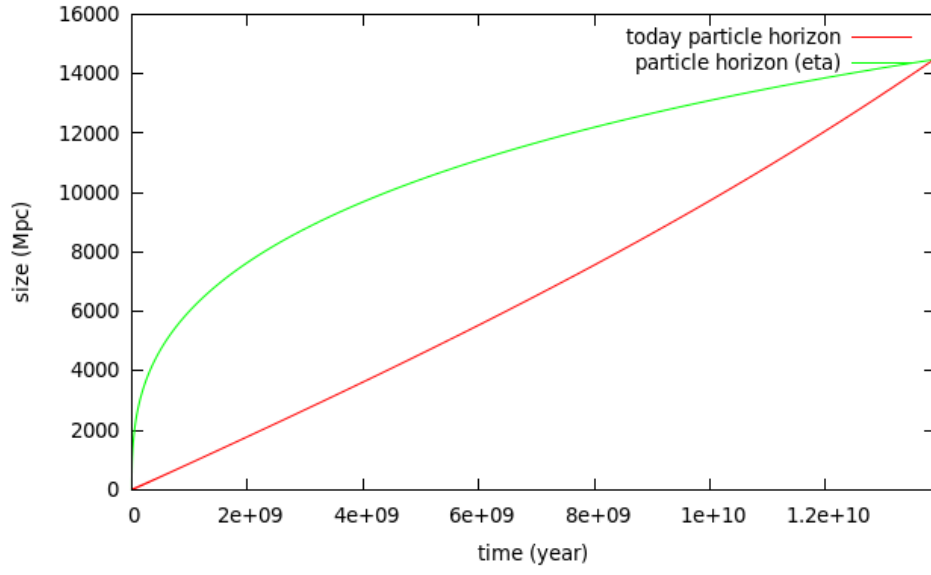


Figure 3.17: evolution of conformal time and today's particle horizon with respect to time for $\alpha = 8$, $\gamma = 0.01$ and $x_{init.} = 1$. The size of today's universe is about $4.36 \times 10^{26} \text{ m} \simeq 14.14 \text{ Gigaparsecs}$.

3.5 The Existence of Two-Field System

The realistic pictures of universe containing two-field and background fluids were found when early universe was dominated by kinetic energy density of u-field, $\frac{1}{2}\dot{u}^2$, and energy density of v-field was small; but according to the data found from numerical solutions, starting from small value of v-field, it becomes more smaller through the time; so v-field does not exist in our universe and two-field system, (u,v), reduces to one field, u, or one scalar field Ψ , defined by:

$$\rho_\Psi = \frac{1}{2}(1 - \gamma^2\kappa^2\Psi^2)\dot{\Psi}^2 + V(\Psi)$$

$$p_\Psi = \frac{1}{2}(1 - \gamma^2\kappa^2\Psi^2)\dot{\Psi}^2 - V(\Psi)$$

with $V(\Psi)$ defined as equation (3.40) and equation of motion defined by (3.7); equation (3.9) will be the same as (3.7).

For this case the relation (3.2), gives:

$$\dot{\varphi} \simeq \gamma\kappa\Psi\dot{\Psi} \quad \text{or} \quad \varphi \simeq \frac{1}{2}\gamma\kappa\Psi^2 \quad (3.53)$$

So graviscalar field φ is dependent on scalar matter field Ψ .

Chapter 4

Quintessence with Hybrid Potential

As discussed in chapter three, power-law potential of scalar matter field Ψ cannot explain inflation; and what is predicted as slow-roll chaotic inflation[18], is in fact accelerated expansion of today's universe, and it can be a good alternative for dark energy. Also graviscalar field, φ is dependent on Ψ and we have only one scalar matter field, Ψ (or u) which plays quintessential role in our universe. But in order to have a universe with both inflation in very early times and accelerated expansion at late times, I want to define a hybrid potential of Ψ , as:

$$V(\Psi) = V_1(\Psi) + V_2(\Psi) = \lambda\kappa^{\alpha-4}\Psi^\alpha + \delta e^{-(\kappa\Psi)^\beta} \quad (4.1)$$

V_1 is the power-law potential defined in chapter three[18]. But V_2 is a exponential potential and it dominates the very early universe. β and δ are constant parameters.

4.1 Universe filled with Background Fluids and Ψ -Field with Hybrid Potential

For a spatially flat universe filled with Ψ -field (with a hybrid potential) and background fluids (dark matter, baryonic matter and radiation), first Friedmann's equation can be written as:

$$H^2 = \frac{\kappa^2}{3} [\pm \frac{1}{2} \dot{u}_\mp^2 + V_1(u_\mp) + V_2(u_\mp) + \rho_r + \rho_m] \quad (4.2)$$

Where again Ψ -field is converted to u -field for simplicity (as defined in chapter three). So density parameters of this system are:

$$x_{\mp}^2 = \frac{\kappa^2 \dot{u}_{\mp}^2}{6H^2} \quad , \quad y_1^2 = \frac{\kappa^2 V_1(u_{\mp})}{3H^2} \quad , \quad y_2^2 = \frac{\kappa^2 V_2(u_{\mp})}{3H^2}$$

$$r^2 = \frac{\kappa^2 \rho_r}{3H^2} \quad , \quad m_d^2 = \frac{\kappa^2 \rho_d}{3H^2} \quad , \quad m_b^2 = \frac{\kappa^2 \rho_b}{3H^2}$$

Where the whole density parameters of matter and potential are:

$$m^2 = m_d^2 + m_b^2$$

$$y^2 = y_1^2 + y_2^2$$

The equation of motion of u -field, for this case is the same as equation (3.12) with hybrid potential density:

$$\ddot{u}_{\mp} + 3H\dot{u}_{\mp} \pm \frac{V_{,\Psi}(u_{\mp})}{\sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_{\mp})|}} = 0 \quad , \quad V_{,\Psi} = V_{1,\Psi} + V_{2,\Psi}$$

And $\frac{\dot{H}}{H^2}$ is: $\frac{\dot{H}}{H^2} = -3 + 3y^2 + r^2 + \frac{3}{2}m^2$

Using these equations and continuity equations of background fluids for adiabatically expanding universe (as discussed in chapter three), the system of autonomous equations, defining the evolution of dimensionless variables and Ψ -field, in terms of logarithm of scale factor, would be found as:

$$x'_{\mp} = -x_{\mp}(3y^2 + r^2 + \frac{3}{2}m^2) + \sqrt{\frac{3}{2}} \frac{\mp \alpha y_1^2 \pm \beta(\kappa\Psi)^{\beta} y_2^2}{\kappa\Psi\sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_{\mp})|}} \quad (4.3)$$

$$y'_1 = y_1(3 - 3y^2 - r^2 - \frac{3}{2}m^2) + \sqrt{\frac{3}{2}} \frac{\alpha y_1 x_{\mp}}{\kappa\Psi\sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_{\mp})|}} \quad (4.4)$$

$$y'_2 = y_2(3 - 3y^2 - r^2 - \frac{3}{2}m^2) - \sqrt{\frac{3}{2}} \frac{\beta(\kappa\Psi)^{\beta} y_2 x_{\mp}}{\kappa\Psi\sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_{\mp})|}} \quad (4.5)$$

$$r' = r(1 - 3y^2 - r^2 - \frac{3}{2}m^2) \quad (4.6)$$

$$m'_d = m_d(\frac{3}{2} - 3y^2 - r^2 - \frac{3}{2}m^2) \quad (4.7)$$

$$m'_b = m_b(\frac{3}{2} - 3y^2 - r^2 - \frac{3}{2}m^2) \quad (4.8)$$

$$\Psi' = \frac{\sqrt{6}x_{\mp}}{\kappa\sqrt{|1 - \gamma^2 \kappa^2 \Psi^2(u_{\mp})|}} \quad (4.9)$$

Numerical solution of this system can be found starting from a very early

point dominated by exponential potential energy ($y_{2, initial} = 1$) or radiation ($r_{initial} = 1$). Figures (4.1) and (4.2) show the evolution of density parameters for these two cases, where $\alpha = 3$ and $\beta = 2$; these pictures are compatible with our universe; and according to these pictures, today's values of energy density parameters are very close to the measured ones [31].

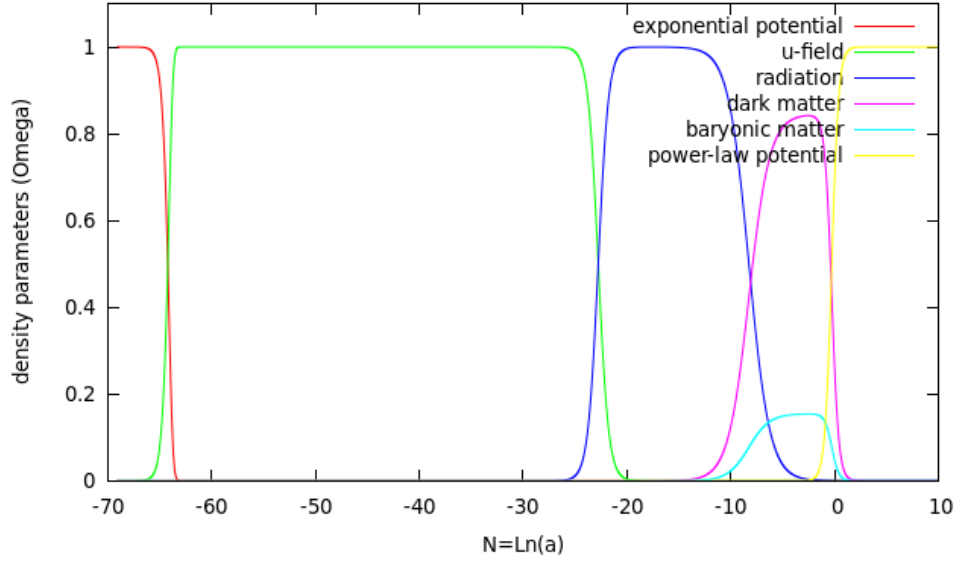


Figure 4.1: Evolution of dynamical variables for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$; starting from a point dominated by exponential potential, $y_{2,init.} = 1$. Today's value of Hubble parameter for this universe is: $H_0 \simeq 2.19 \times 10^{-18} S^{-1}$ and the age of the universe is: $\sim 4.33 \times 10^{17} S$. Dark energy, dark Matter and baryonic matter density parameters of today's universe are: $\Omega_\Lambda \simeq 0.687$, $\Omega_{d,m} \simeq 0.263$ and $\Omega_{b,m} \simeq 0.048$.

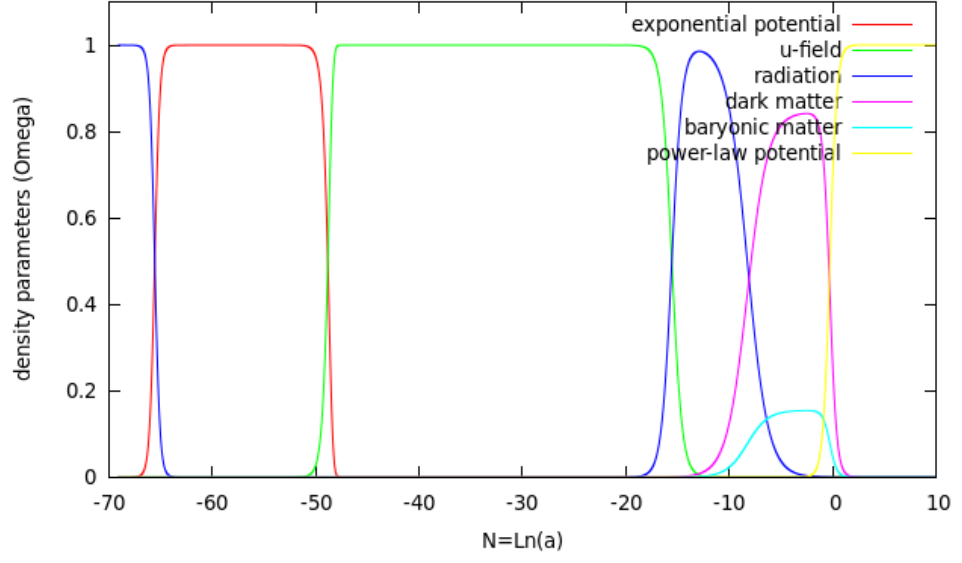


Figure 4.2: Evolution of dynamical variables for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$; starting from a point dominated by radiation, $r_{init.} = 1$. Today's value of Hubble parameter for this universe is: $H_0 \simeq 2.18 \times 10^{-18} S^{-1}$ and the age of the universe is: $\sim 4.35 \times 10^{17} S$. Dark energy, dark Matter and baryonic matter density parameters of today's universe are: $\Omega_\Lambda \simeq 0.683$, $\Omega_{d,m} \simeq 0.267$ and $\Omega_{b,m} \simeq 0.0489$.

4.2 Potential and Cosmological Parameters

Variation of Potential Energy Density with κu

Figures (4.3) to (4.6) show the variation of power-law potential $V_1(\kappa u)$ and exponential potential $V_2(\kappa u)$, for $\alpha = 3$ and $\beta = 2$. The behavior of the power-law potential, V_1 is the same for both the universe started from exponential potential dominated or radiation dominated era; V_1 increases with u and time, and it's going to dominate the future's universe. Exponential potential V_2 falls down from a very high value and decays rapidly, in the very early universe; but when the very early universe is dominated by exponential potential, the initial density of exponential potential is about $4.18 \times 10^{117} \text{ kg/m}^3$, that is very higher than its value ($\simeq 5.7 \times 10^{83} \text{ kg/m}^3$) when the very early universe is dominated by radiation.

Power-law potential V_1 is a good alternative for dark energy; its density parameter evolves in our universe similar to the dark energy and its today's density is very close (or the same as) observed value of dark energy density of today's universe[31].

Exponential potential, V_2 which provides negative pressure for our early universe, may be an explanation of inflation.

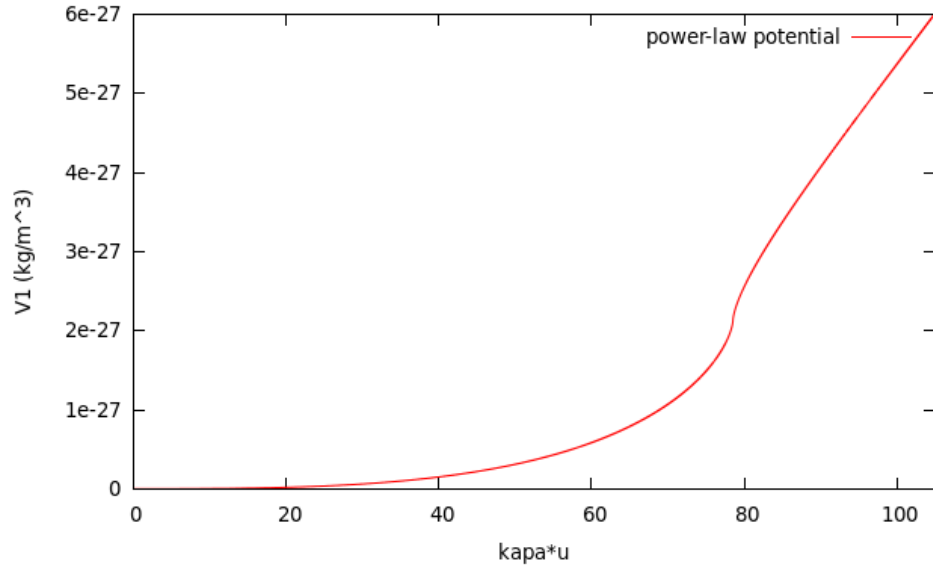


Figure 4.3: Variation of power-law potential energy density with κu , found for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from exponential potential dominated era, $y_{2,initial} = 1$. Today's value of power-law potential is: $V_{1,today} \simeq 5.94 \times 10^{-27} \text{ kg/m}^3$.

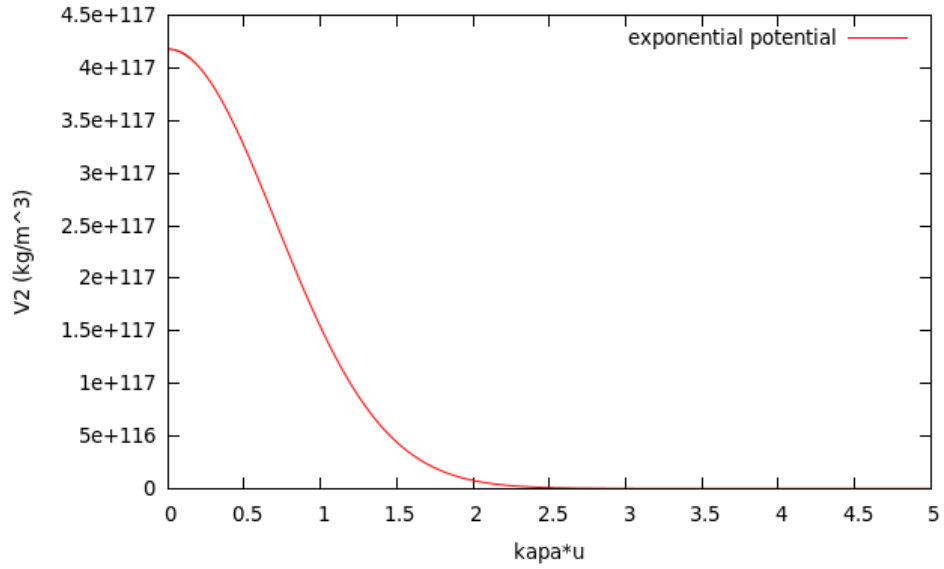


Figure 4.4: Variation of exponential potential energy density with κu , found for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from exponential potential dominated era, $y_{2,initial} = 1$.

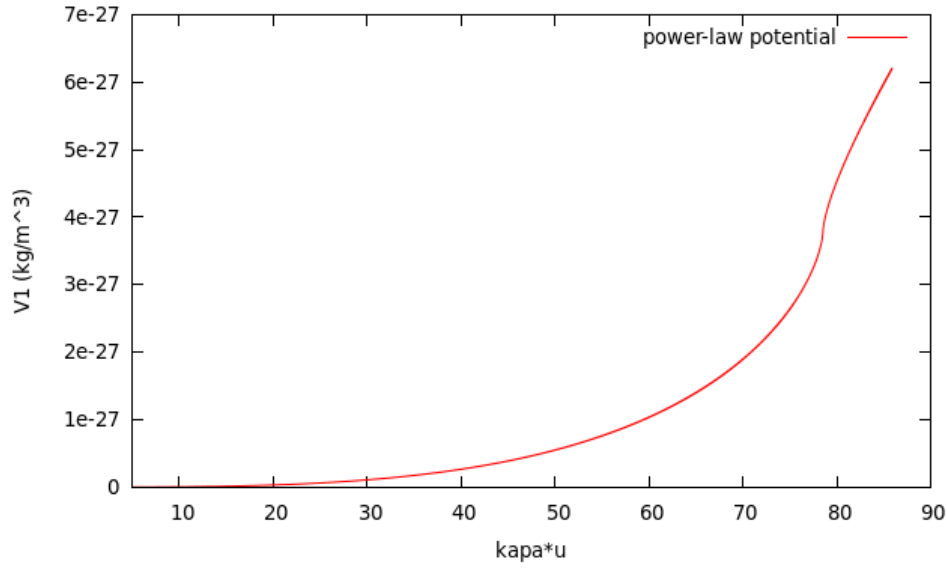


Figure 4.5: Variation of power-law potential energy density with κu , found for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from radiation dominated era, $r_{\text{initial}} = 1$. Today's value of power-law potential is: $V_{1,\text{today}} \simeq 5.8 \times 10^{-27} \text{kg/m}^3$.

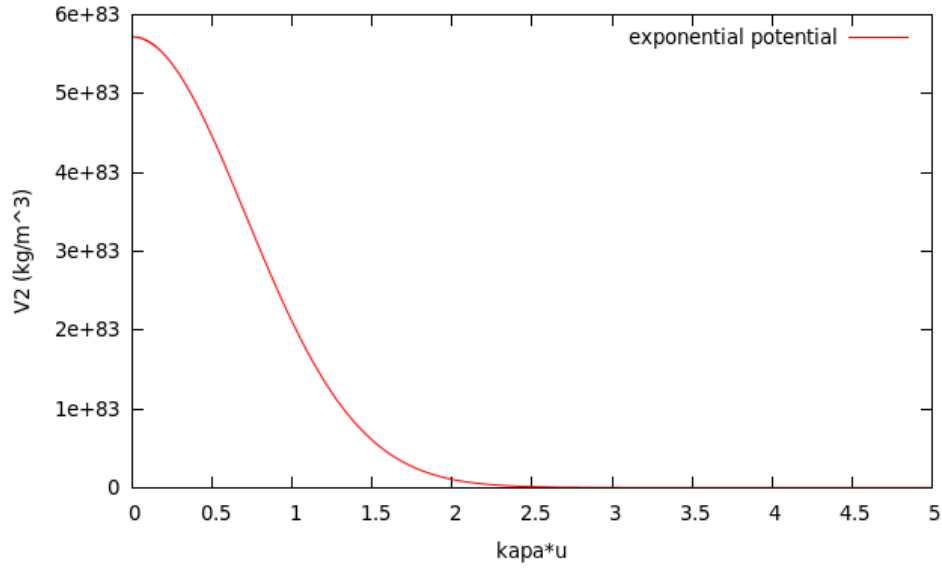


Figure 4.6: Variation of exponential potential energy density with κu , found for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from radiation dominated era, $r_{\text{initial}} = 1$.

Equation of State

Equation of state of our universe filled with background fluids and scalar matter field, Ψ with hybrid potential energy, can be found as:

$$W = \frac{P}{\rho} = \frac{\pm \frac{1}{2} \dot{u}_{\mp}^2 - V + \frac{1}{3} \rho_r}{\pm \frac{1}{2} \dot{u}_{\mp}^2 + V + \rho_r + \rho_m}$$

$$\Rightarrow W = 1 - 2y_1^2 - 2y_2^2 - m_d^2 - m_b^2 - \frac{2}{3}r^2 \quad (4.10)$$

Figures (4.7) and (4.8) found for $\alpha = 3$, $\beta = 2$, show how the equation of state vary, as our universe evolves from very early stages to the future.

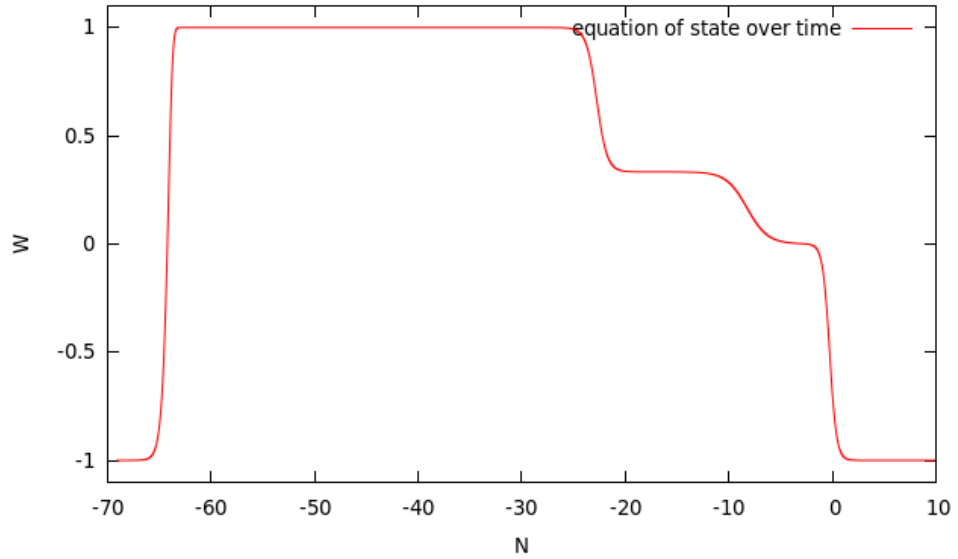


Figure 4.7: Evolution of equation of state for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from exponential potential dominated era, $y_{2,initial} = 1$.

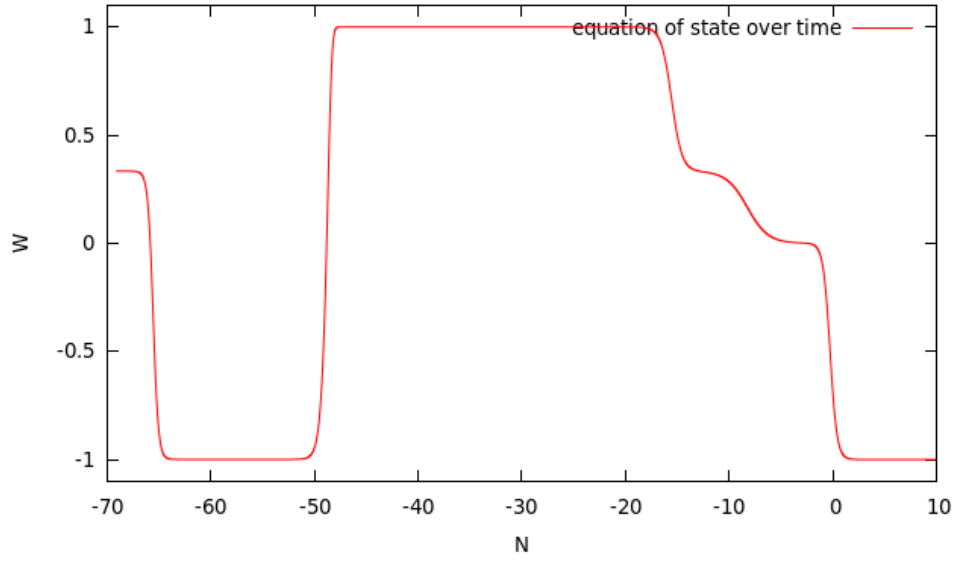


Figure 4.8: Evolution of equation of state for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from radiation dominated era, $r_{initial} = 1$.

Hubble Parameter

According to the pictures (4.9) and (4.10), Hubble parameter of very early universe was very high and during the domination of exponential potential it had very small increase; so it seems constant at that era. When positive pressure of u-field and then radiation, dominates our universe, Hubble parameter decreases; but because of the negative pressure of power-law potential, it's going to increase in the future, after $N=2$, as shown in figure (4.11). Today's values of Hubble parameter found for two cases (starting from exponential potential or radiation), are the same as its observed value [31].

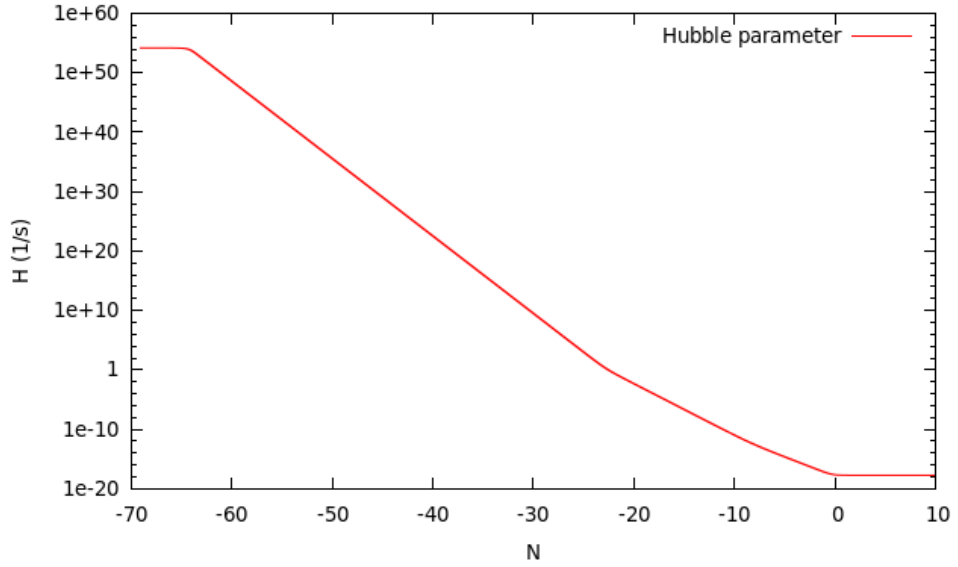


Figure 4.9: Evolution of Hubble parameter for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from exponential potential dominated era, $y_{2,initial} = 1$. Today's value of Hubble parameter for this case is: $H_0 = 2.19 \times 10^{-18} \text{ s}^{-1} \simeq 67.82 \text{ km/s/Mpc}$.

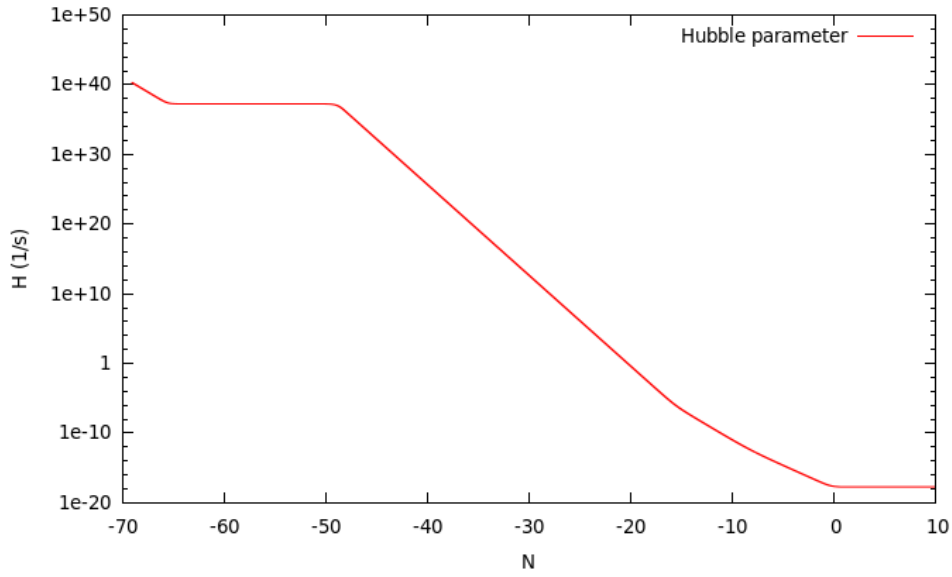


Figure 4.10: Evolution of Hubble parameter for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from radiation dominated era, $r_{initial} = 1$. Today's value of Hubble parameter for this case is: $H_0 = 2.18 \times 10^{-18} s^{-1} \simeq 67.36 km/s/Mpc$.

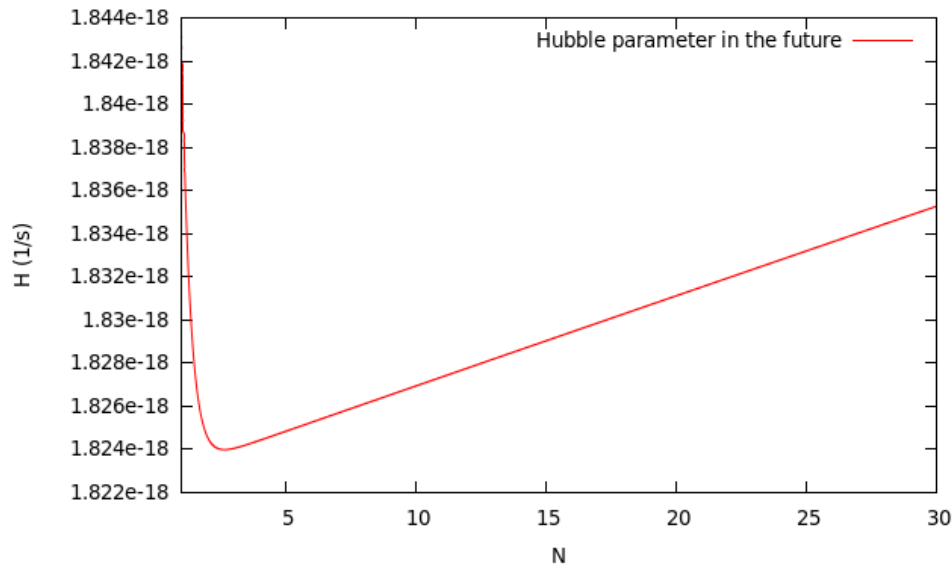


Figure 4.11: Evolution of Hubble parameter in the future, for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$.

Size and Age of the Universe

The size of our today's visible universe over time and the particle horizon of universe at any given time (or conformal time) are shown in pictures (4.12) and (4.13) for two different cases; according to these pictures our visible universe has always been inside particle horizon in the past; but in the future the most distant parts of it will go to the outside of the particle horizon, gradually.

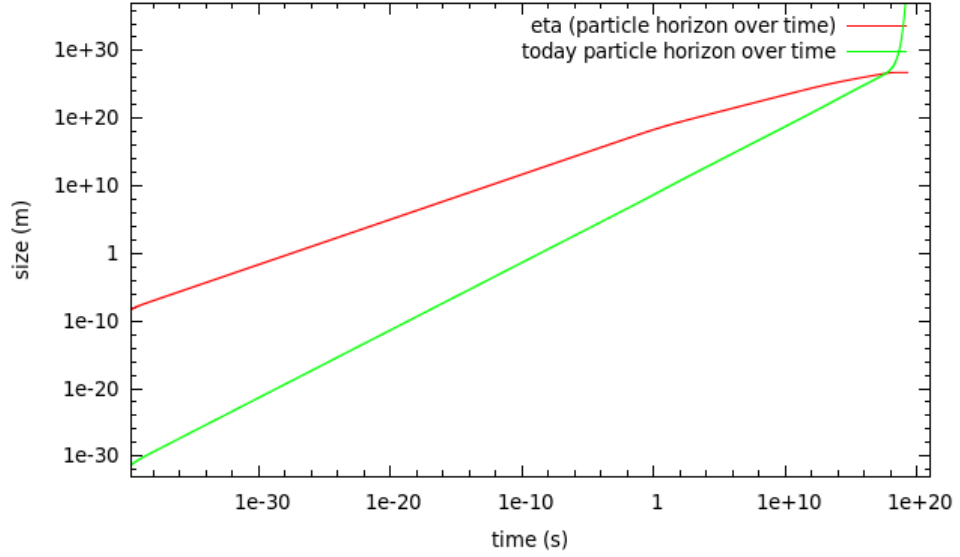


Figure 4.12: Evolution of conformal time and today's particle horizon over time for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from exponential potential dominated era, $y_{2,initial} = 1$. The size of today's universe is about $4.35 \times 10^{26} \text{ m} \simeq 14.12 \text{ Giga parsecs}$; and its age is about $4.33 \times 10^{17} \text{ s} \simeq 13.7 \text{ Giga year}$.

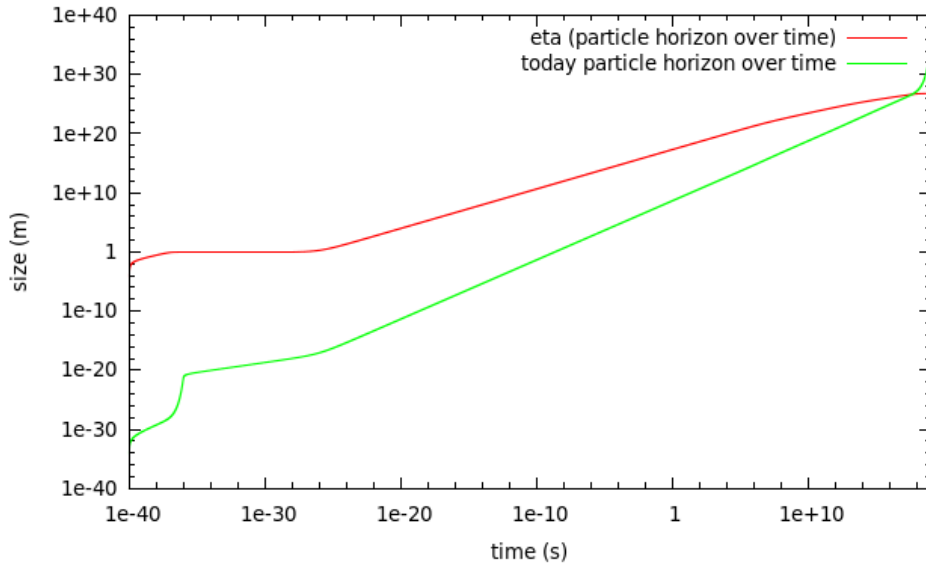


Figure 4.13: Evolution of conformal time and today's particle horizon over time for $\alpha = 3$, $\beta = 2$, $\gamma = 0.01$, when our universe starts from radiation dominated era, $r_{initial} = 1$. The size of today's universe is about $4.36 \times 10^{26} \text{ m} \simeq 14.141 \text{ Giga parsecs}$; and its age is about $4.349 \times 10^{17} \text{ s} \simeq 13.7 \text{ Giga year}$. According to the picture, when the horizon of the inflationary universe is about 1 meter, the size of our visible universe is negligible. And when the size of our visible universe is about 1 meter, the particle horizon is about 10^{15} meters; the age of the universe is about 10^{-7} seconds at that time.

Today's Values of Cosmological Parameters

For the two cases as discussed before, some important cosmological parameters of today's universe are shown in the table below:

Cosmological Parameters	starting from Exponential Potential domination	starting from Radiation domination
Hubble parameter, H_0	$2.19 \times 10^{-18} \text{ s}^{-1}$	$2.18 \times 10^{-18} \text{ s}^{-1}$
size, η_0	$4.35 \times 10^{26} \text{ m}$	$4.36 \times 10^{26} \text{ m}$
age, t_0	$4.336 \times 10^{17} \text{ s}$	$4.349 \times 10^{17} \text{ s}$
potential energy density, V_0	$5.94 \times 10^{-27} \text{ kg/m}^3$	$5.8 \times 10^{-27} \text{ kg/m}^3$
$\Omega_{\Lambda,0}$	0.687	0.683
$\Omega_{dm,0}$	0.263	0.267
$\Omega_{bm,0}$	0.048	0.0489
$\Omega_{r,0}$	8.17×10^{-5}	8.28×10^{-5}

Chapter 5

Conclusions

In this thesis I studied two models of quintessence: scalar field and non-metric quintessence; and for both of them, I found the numerical solutions of equations of motion.

According to the phase plots of scalar field with exponential potential, our universe can be in equilibrium points, dominated by scalar field or background fluids, or both of them (scaling solutions); although the stability and fluid contents and the conditions of existence of equilibrium points were studied precisely, but these plots do not give entire information about the evolution of the universe. So in order to find if scalar field with exponential potential, is the correct model of quintessence, I solved the equations of motion of dimensionless variables defining the evolution of universe filled with scalar field, radiation and matter (dark and baryonic matter), numerically. The plots of evolution of density parameters and other cosmological parameters found from the numerical solutions, are not compatible with real model of universe and measurements. So scalar field with exponential potential cannot give a realistic model of our universe.

Non-metric quintessence was the second model of dark energy studied in this thesis. This model is a combination of two coupled fields (Ψ, φ) ; Ψ is a scalar matter field and φ is a graviscalar field. After decoupling them and defining new variables, we have a system of two-field defined by (u_{\mp}, v) ; scalar field u_{\mp} is defined for two regions of validity of Ψ and with its power-law potential play a quintessential role in our universe. Dynamics of dimensionless variables defining this two-field system make a non-autonomous system; so we cannot get any information about our universe from the phase diagrams of scalar field, u_{\mp} , except the symmetries of the system. So in order to find the evolution of a universe filled with this two-field system and background fluids (radiation, dark matter and ordinary matter), I solved the equations of dynamics of dimensionless variables defining these fluids, numerically. If early universe was dominated by v-field, the numerical solutions are not compatible with observations and the real history of our universe.

But starting from a very early universe dominated by kinetic energy of scalar field, $\frac{1}{2}\dot{u}^2$, the evolution of density parameters are entirely compatible to the real picture of the history of our universe, and the size and age of the universe and today's values of Hubble parameter, radiation and matter densities, found from numerical solutions, are the same as observational and measured values. According to the results, the potential energy, $V(u)$, which provides negative pressure, is going to dominate our universe in the future, and its today's density parameter is the same as the density of the dark energy in our universe. Although these results are compatible with our universe, but they show that v-field does not exist in our universe; so two-field system reduces to only one scalar matter field; the potential energy of this field does not explain inflation; and what was supposed to be slow-roll chaotic inflation[18], is infact the accelerated expansion of today's universe. And the kinetic energy of scalar matter field with its positive pressure, slows down the expansion of very early universe.

In order to find an explanation for early inflationary universe, I defined a hybrid potential of scalar matter field Ψ , that is a combination of a power-law potential (as defined before) and an exponential potential (gaussian) of Ψ -field. The exponential potential energy dominates very early universe and it can explain inflation, and power-law potential energy is again a good alternative for dark energy.

The results found from the numerical solutions of the system of equations of dimensionless variables defining the evolution of a universe filled with background fluids (dark matter, baryonic matter and radiation) and scalar matter field Ψ (with hybrid potential), are consistent with real universe. The plot of energy density parameters gives the real history of our universe; and today's values of density parameters of dark energy, dark matter, baryonic matter, and Hubble parameter, and the age and size of our universe, found from this model, are very close to (and some times the same as) measured values.

I've found the numerical solutions for two cases: the very early universe dominated either by exponential potential or by radiation. Starting from radiation domination, exponential potential appears later in our early universe and causes inflation; this case is more compatible with our early universe.

Appendix A

Numerical Methods

In order to solve autonomous system of equations (in chapter two through four) and find the evolution of density parameters, Ψ -field, and the other cosmological parameters (Hubble parameter, age, size, and today's particle horizon), Bulirsch-Stoer method has been used in this project. Bulirsch-Stoer method [32] is a numerical approach to differential equations which cannot be solved analytically; it starts from a point where all the parameters are known and by using their derivatives, calculates their values at the next point. So the values of the parameters can be found at all the grid points, step by step; and for the large number of grid points, we can find the more correct answers. But the scalar matter field, Ψ exists in two different regions (minus and plus) and we have two different set of autonomous system of equations valid in these two different regions. The numerical calculations start from the minus region and the numerical solutions of the parameters are found at the grid points, $N=\text{Ln}(a)$; so at the last point of the minus region, $N_{-,f}$, all the parameters are found; but at $(N_{-,f} + 1)$, the corresponding parameters cannot be found either in minus region or in plus region, because $\Psi(N_{-,f})$ belongs to the minus region and $\Psi(N_{-,f} + 1)$ belongs to the plus region. So at the point $(N_{-,f} + 1)$, I decided to calculate Ψ , from this relation:

$$\Psi(N_{-,f} + 1) = \Psi(N_{-,f}) + [\Psi(N_{-,f}) - \Psi(N_{-,f} - 1)]$$

I did it for the other parameters of $(N_{-,f} + 1)$, too. So at the point $(N_{-,f} + 1)$ that is the begining point of the plus region, all the variables are known; and Bulirsch-Stoer method uses them as initial values and calculate the values of the parameters at the next points, step by step. The crashing of the Bulirsch-Stoer method at the transition point causes deviations in numerical calculations and because of that most of the plots are not smooth at the transition point.

In chapter three the phase plots of non-autonomous system of equations of two-field, (x', y') are also found numerically. I started from a point where

$x=y=0$, so $x' = 0$ and $y' = 0$, and by using these two relations:

$$y' = y'(x, y, x')$$

$$x' = x'(x, y, y')$$

I found the values of x' and y' at next points, step by step by using previous values of x' and y' .

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